LINES OF CONSTANT GEODESIC CURVATURE AND THEIR $\underline{k}$-FLOWS.

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june 1974.

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Preface:

This thesis was prepared while I was a research student at Warwick during the years 1970-1973. I should like to thank the people I met during my stay for their encouragement,ideas and in general making it a very pleasant three years; in particular my supervisor, Peter Walters, for directing and criticising my research activities, and being an excellent mathematical mentor. Finally the Science Research Council for their financial support.

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June 1974.

Abstract.

Much work has been done on the geodesics of a Riemannian manifold and the flow it induces on the unit tangent bundle, particularly on manifolds of negative curvature. It is the purpose of this thesis to extend this work to a more general case, considering those curves of a manifold with constant geodesic curvature.

In the first chapter we define these k-lines and develop some ideas about their geometry, contrasting and comparing them with the geodesics. We show how they give flows on T^1M, develop a variational theory for them and show how matrix methods may be used to solve the variational equations.

In the second chapter we investigate a particular property, that of Anosovity, well known and documented in the geodesic case. For a particular class of k-flows we solve the matrix variation equations, giving necessary and sufficient conditions for Anosovity in terms of the geodesic normals and curvatures. On manifolds of negative curvature we assign an 'Anosov' number to each flow such that if it is less than zero the flow is Anosov. We end by considering families of k-flows and for a class of flows indicate the topological similarities between members of the class and in particular the geodesic flow. We end with some conjectures and ideas for future work.

Also included are two appendices. The first catalogues the results we need on the metric of the unit tangent bundle, and we extend this to the Frame bundles

defining <u>k</u>-flows on these spaces and investigating the volume preserving properties. The second derives <u>k</u>-lines and flows from a classical mechanics point of view and justifies their study as phenomena arising from physical situations.

CONTENTS.

Chapter 0: Notation and Terminology.

The main techniques of this thesis are concerned with the study of differential geometric properties of a class of curves in a Riemannian manifold, M, by treating the variation vector fields generated by the curves through matrix theory. Below we shall catalogue the notation and terminology to be used subsequently, and also some known results which will be used at several places in the text. The differential geometry of the unit tangent bundle we have considered seperately in Appendix A, since a generalisation of the results to Frame Bundles, though not directly relevant to the main body of the thesis is of some independent interest and is covered there.

First let us consider the differential geometry that we shall need:-

Let M be a C^{∞} Riemannian manifold with metric tensor, g. By TM we shall mean the tangent bundle of M, and if $f:M \longrightarrow N$ is a differentiable manifold map then $Tf:TM \longrightarrow TN$ is the usual induced map.

The higher tangent bundles and their projections are given by the sequence:-

$$T^rM \xrightarrow{\pi_r} T^{r-1}M \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} T^2M \xrightarrow{\pi_1} TM \xrightarrow{\pi} M$$

The metric tensor, g, induces an inner product $\langle\ ,\ \rangle_x$ and a norm $\|\ \|_x$, on each fibre T_xM. Under this norm let $T^1M = \left\{ v \in TM : \|v\| = 1 \right\}$ be the Unit Tangent (Sphere) Bundle.

If $\mathfrak{X}(M)$ is the module of vectorfields on M and $\mathfrak{F}(M)$ the set of smooth functions $: M \longrightarrow \mathbb{R}$, then there is a unique (torsion-free) symmetric connexion, ∇, satisfying:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) \quad \text{s.t.}$$

(i) $\nabla_{X_1+X_2}Y = \nabla_{X_1}Y + \nabla_{X_2}Y$, $\nabla_{X_1}(Y_1+Y_2) = \nabla_X Y_1 + \nabla_X Y_2$.

for all $X,Y,X_1,X_2,Y_1,Y_2 \in \mathfrak{X}(M)$.

(ii) $\nabla_{fX}Y = f\nabla_X Y$. for $f \in \mathfrak{F}(M)$, $X,Y \in \mathfrak{X}(M)$.

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$. for $f \in \mathfrak{F}(M)$, $X,Y \in \mathfrak{X}(M)$

and that is compatible with g.

By $[.\ ]$ and Γ we shall indicate the Christoffel symbols of the first and second kind of this compatible metric.

The following results will be used:-

<u>Lemma 0:1.</u> (Spivak[25]p6:15)

∇ is compatible with g iff given any vectorfields X,Y along any curve $c: I \longrightarrow M$ then

$$d/_{dt}\langle X,Y\rangle = \langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$$ where v is the velocity $\dot{c}$ along the curve.

Alternatively if $c: I \longrightarrow M$ is a curve in M and $X(t) = X(c(t))$ a vectorfield along c we shall sometimes denote $\nabla_{v(t)}X$ by $\frac{DX}{dt}$, i.e. the covariant derivative of X along c.

Let R be the Riemann Curvature Tensor.

i.e. $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ by

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

<u>Lemma 0:2</u> (Spivak[25]p6:27)

For all $X,Y,Z,W \in \mathfrak{X}(M)$ then (i) $\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$

(ii) $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$.

<u>The Frame Bundles:</u>

By $F^{r+1}M$ we shall represent the principal bundle over M consisting of all ordered orthonormal (r+1) frames $\{(x,\xi_1,\dots,\xi_{r+1}) \in TM \oplus \dots \oplus TM \text{ s.t. } \langle \xi_i,\xi_j\rangle = \delta^i_j\}$; regarding

T^1M as F^1M.

In Appendix A we consider TT^1M and $TF^{r+1}M$ showing that g induces a Riemannian metric G, on each of them, with inner products $\langle\!\langle\ ,\ \rangle\!\rangle$ and norms $\|\!|\ \ \|\!|$ on fibres, and Christoffel symbols $[\![\ \ ]\!]$ and $\Gamma\!\!\Gamma$.

If (U,φ) is a co-ordinate chart on M then $(TU,T\varphi)$, $(T^2U,T^2\varphi)$ etc. can be taken as co-ordinate charts on TM, T^2M etc. These co-ordinate charts use ordinary differentials, whereas in Appendix A we develop other co-ordinate systems for T^2M, TT^1M and $TF^{r+1}M$ based on covariant differentiation.

<u>Notation</u>.

'Natural Co-ordinates' for $T^2M : \{(x,v,\xi_1,\xi_2) \in T^2U.\}$

'Covariant Co-ordinates' " " $: \{(x,v,\ e_1 \oplus e_2) \in T^2U.\}$

Similarly for $TF^{r+1}M$:

'Natural Co-ordinates' for $TF^{r+1}M . \{(x,v,\xi_1,\ \ldots,\xi_r) \in TF^{r+1}U\}$

'Covariant Co-ordinates' " " $: \{(x,v,e_1 \oplus \ldots \oplus e_r) \in TF^{r+1}U\}$

In particular the connector map $K:T^2M \longrightarrow TM$ for the connexion ∇, (Eliasson [10]), takes the simple form of a projection in covariant co;ordinates:

$$K:T^2U \longrightarrow TU : (x,v,e \oplus f) \longrightarrow (x,f).$$

This complemements the induced projection:

$$T\pi_1 : T^2U \longrightarrow TU : (x,v,e \oplus f) \longrightarrow (x,e).$$

For further details see Appendix A.

<u>Variation Surfaces.</u>

When considering variation theory of flow lines we shall need:

A (parameterised) <u>Surface s in a manifold M</u> is a C^∞

function $s : I \times I \longrightarrow M : (u,t) \longrightarrow s(u,t)$.

A vectorfield V on s is a C^∞ function $V : I \times I \to TM$ such that $V(u,t) = V(s(u,t)) \in T_{s(u,t)}M$.

In particular, in analogy to the velocity of a curve, we have two standard vectorfields on a surface s, namely

$$\frac{\partial s}{\partial u} = s_*(\partial/\partial u) \quad \text{and} \quad \frac{\partial s}{\partial t} = s_*(\partial/\partial t).$$

For fixed t_0 by condidering the curve $s(u,t_0)$ we have the covariant derivative; $\left.\frac{DV}{du}\right|_{t=t_0}$, of V. Similarly $\left.\frac{DV}{dt}\right|_{u=u_0}$.

Lemma 0:3. (Spivak [25] pp6:18,6:20).

If V is a vectorfield on a parameterised surface s then:

(i) $\frac{D}{\partial u}\left(\frac{\partial s}{\partial t}\right) = \frac{D}{\partial t}\left(\frac{\partial s}{\partial u}\right)$

(ii) $\frac{D}{\partial u}\frac{DV}{\partial t} - \frac{D}{\partial t}\frac{D}{\partial u} = R\left(\frac{\partial s}{\partial t};\frac{\partial s}{\partial u}\right)V.$

Volumes on Manifolds.

By an volume Ω on a manifold, M, we mean an n-form on M such that $\Omega(m) \neq 0$ for all $m \in M$. i.e. $\Omega(m)$ is a skew-symmetric map $\Omega(m) : T_mM \times \ldots \times T_mM \longrightarrow \mathbb{R}$.

Given a Riemann manifold, M, we have a natural volume element Ω_0 for if (U,φ) is a chart on M with co-ordinates $(x_1,\ldots x_n)$ then $\Omega_0(u) = (dx^1(u) \wedge dx^2(u) \wedge \ldots \wedge dx^n(u)) / \sqrt{\det g_{ij}}$

If X is a vectorfield on M then

$$\operatorname{div}_{\Omega_0} X = \frac{\partial X^i}{\partial x^i} + \Gamma^i_{jj}(x)X^j.$$

This is a special case of the divergence of a vector field X, on a manifold M, w.r.t. an arbitary volume Ω on M (Abraham [1])

By the use of moving frames along flow lines we convert facts about vectorfields into equations in real $(n \times n)$ matrices.

<u>Lemma 0:4</u> If A is a real symmetric $n \times n$ matrix then there is an orthogonal matrix O such that OAO^T is diagonal with entries the eigenvalues of A. ([21] p404)

By use of series we can show that we can define:-

$$\exp(tA) = I + At + \tfrac{1}{2}A^2t^2 + \tfrac{1}{3}A^3t^3 \ldots\ldots$$

$$\sin(tA) = \tfrac{1}{2i}\left(\exp(itA) - \exp(-itA)\right) = At - \tfrac{1}{3}t^3A^3 + \tfrac{1}{5}t^5A^5 \ldots$$

$$\cos(tA) = \tfrac{1}{2}\left(\exp(itA) + \exp(-itA)\right) = I - \tfrac{1}{2}t^2A^2 + \tfrac{1}{4}t^4A^4 \ldots\ldots$$

We shall be looking at the norms $\| \ \|$, of vectors $v \in TM$, and find that it is closely related to one of the standard matrix norms. We shall also have to use a variety of matrix norms:

(0) Euclidean norm $|A_{ij}| = (\sum_{i,j} A_{ij}^2)^{\frac{1}{2}}$

(1) $|A_{ij}|_1 = (\sum_{i,j} |A_{ij}|)$

(2) Largest column norm $|A_{ij}|_2 = \max_j (\sum_i A_{ij}^2)^{\frac{1}{2}}$

(3) $|A_{ij}|_3 = \sup_{|x|=1} |Ax|$ for column matrices x ; the Operator norm.

(4) $|A_{ij}|_4 = \max_{j=1,\ldots,n} (\sum_{i=1}^{n} |A_{ij}|)$

(5) $|A_{ij}|_5 = \max_{i,j=1,\ldots,n} |A_{ij}|$

Fortunately we have a means of comparing these norms:

<u>Lemma 0:5</u> ([17]) All matrix norms are equivalent.i.e. given two matrix norms $| \ |_i$ and $| \ |_j$ then there are constants a and b such that

$$a|A|_i \leqslant |A|_j \leqslant b|A|_i, \quad a,b>0$$

In particular for $n \times n$ matrices:

$$\tfrac{1}{n}|A|_1 \leqslant |A| \leqslant |A|_1, \quad \tfrac{1}{n^{3/2}}|A|_1 \leqslant |A|_2 \leqslant |A|_1,$$

$$\tfrac{1}{n}|A|_1 \leqslant |A|_3 \leqslant |A|_1, \quad \tfrac{1}{n}|A|_1 \leqslant |A|_4 \leqslant |A|_1,$$

$$\tfrac{1}{n^2}|A|_1 \leqslant |A|_5 \leqslant |A|_1.$$

Similarly for column matrices:

$$|A|_1 = |A|_4 \,;\, |A| = |A|_1 \,,\, \frac{1}{n^{1/2}}|A|_1 \leqslant |A|_5 \leqslant |A|_1 .$$

Lemma 0:6 (1) If O is an orthogonal matrix $|O|_2 = 1$ [p400 [21]]

(2) If M and N are orthogonally equivalent ($\exists$ O orth. s.t. $OMO^T = N$) then $|N|_2 = |M|_2$

(3) If A is real and symmetric then $|A|_2 = \{\max_i |\lambda_i| : \lambda_i \text{ e.v. of } A\}$

(4) If A is an $n \times n$ matrix and x an n-column matrix then $|Ax|_i \leqslant |A|_3 |x|_i$ considering A as an operator.

In addition we use the following abbreviations:-

st	such that
iff	if and only if
nhd	neighbourhood
$\in$	belongs to
p-w	piece-wise
I	$[0,1]$
/	restricted to
A^T	transpose of matrix A
A^{-T}	$(A^{-1})^T$

CHAPTER ONE: THE DEFINITION AND VARIATION THEORY OF k-FLOWS ON MANIFOLDS.

Introduction. Given a Riemannian manifold, M, with metric tensor, g, and compatible connexion, ∇, we can first ask what the straight lines, the geodesics, are; this being well documented both classically and in the modern setting. In particular the notion that a particle in a plane with no external forces acting moves in a straight line has been generalised to manifolds. (Abraham[1]). On a Riemannian manifold we can define the Kinetic Energy of a particle at $m \in M$, travelling with velocity $v \in T_mM$ as $\frac{1}{2}g(m)(v,v)$. The idea of the Lagrangian, L, has also been generalised, and if we take L as the Kinetic Energy then the particles move along the geodesics of the manifold, (Abraham[1]) Globally the situation is studied as a flow on T^1M, the unit tangent bundle, the trajectories of which are the canonical lifts of the geodesics of M. This reflects the fact that through any point and for any initial direction we have a unique geodesic that a particle can follow. The properties of this flow have been well documented over the last few years.

Our intention is to generalise these ideas and define a wider family of flows. Reverting to particles on a plane let us now consider a magnetic field perpendicular to the plane. The particles then move in arcs of circles, being bent in a direction given by Fleming's rule.

We should like to generalise these ideas to a Riemann manifold and study the flows they generate, keeping the above example in mind.

Section 1. Lines of Geodesic Curvature.

Given a Riemannian manifold, (M,g,∇), and a curve $c:I\longrightarrow M$, then through each point $c(t)$ on c there is a velocity element $v(t) = c'(t) = Tc(t,1)$, and c is a geodesic iff $\nabla_{v(t)} v = 0$ for all t.

Now $\nabla_v v = 0 \Rightarrow d/dt \langle v(t), v(t)\rangle = 2\langle v, \nabla_v v\rangle = 0$.
So c has a constant speed and if parameterised by arc length then $\|v(t)\| \equiv 1$.

Setting $T(t) = v(t)/\|v(t)\|$ then a geodesic is a curve such that $\|v(t)\|$ = constant and $\nabla_T T = 0$. Hence the geodesics through $m\in M$ with initial velocities v and constant$\times v$ are the same lines in M, differing only in speed.

Consider the case of curves in M with $\|v(t)\|$ a constant but $\|\nabla_T T\| \equiv k_1 \geqslant 0$.

(1) If $k_1 = 0$ then we get geodesics.

(2) If $k_1 \neq 0$ then $\langle T,T\rangle$ = constant implies $2\langle\nabla_T T, T\rangle = 0$
So $\nabla_T T$ is perpendicular to T and we can set $\nabla_T T = k_1 N_1$ for some unit vector N_1 normal to T.

k_1 is the (1st) GEODESIC CURVATURE and N_1 the (1st) GEODESIC NORMAL.

(3) Now $(\nabla_T N_1 + k_1 T)$ is perpendicular to T and N_1.
Let $\|\nabla_T N_1 + k_1 T\| = k_2$ be a constant, so $\nabla_T N_1 + k_1 T = k_2 N_2$.
k_2 is the 2nd GEODESIC CURVATURE(GEODESIC TORSION) and N_2 the 2nd GEODESIC NORMAL.

Similarly $\nabla_T N_2 + k_2 N_1$ is perpendicular to T, N_1 and N_2, and we can define higher curvatures and normals. However M being n-dimensional and $T, N_1, \ldots, N_r$ being mutually perpendicular means that there must be some normal N_r, $r \leqslant n-1$, such that $\nabla_T N_r + k_r N_{r-1} = 0$.

Notation. Let $\underline{k}$ be an (n-1)tuple of numbers $k_i \geqslant 0$, $(k_1, \ldots, k_{n-1})$, such that if $k_j = 0$ then $k_{i+j} = 0$ for all $i > 0$.
i.e. $\underline{k} = (3,2,0,0,\ldots,0)$ not $(3,0,2,\ldots,0)$

<u>Definition</u>: A curve $c: I \to M$ is a $\underline{k}$-line, if it is a curve of geodesic curvatures $k_1, \dots k_r$ where $\underline{k} = (k_1, \dots, k_r, 0, \dots 0)$.
i.e. (i) constant $= \|c'(t)\|$ so $c'(t) = v(t) = \|v(t)\|\, T(t)$.

(ii) For each t there are unit vectors $N_1, \dots, N_r \in T^1_{c(t)}M$ s.t. $(T(t), N_1(t), \dots, N_r(t))$ are mutually orthogonal and

$$A\begin{cases}(0)\ \nabla_{T(t)}T = k_1 N_1(t).\\ (1)\ \nabla_{T(t)}N_1 = k_2 N_2(t) - k_1 T(t).\\ (2)\ \nabla_{T(t)}N_2 = k_3 N_3(t) - k_2 N_1(t).\\ \quad\vdots\\ (i)\ \nabla_{T(t)}N_i = k_{i+1}N_{i+1}(t) - k_i N_{i-1}(t).\\ \quad\vdots\\ (r)\ \nabla_{T(t)}N_r = -k_r N_{r-1}(t).\end{cases}$$

$\{T(t), N_1(t), \dots, N_r(t)\}$ are <u>FRENET VECTORS</u> and A above <u>FRENET FORMULAE</u>, analogous to the Serret-Frenet formulae for curves in $\mathbb{R}^3$, since when $k_3=0$ we have;

$$\nabla_T T = k\,N$$
$$\nabla_T N = \tau B - kT$$
$$\nabla_T B = -\tau N$$

Notes: (1) A geodesic is a $\underline{0}$-line. $\underline{0} = (0,0,\dots,0)$

(2) We have restricted attention to $\underline{k}$ = constant. We could have had $\underline{k} = \underline{k}(m)$ for $m \in M$, corresponding to a magnetic field of variable field strength.

<u>Geometric Properties of $\underline{k}$-lines.</u>

Literature on $\underline{k}$-lines is not too extensive. Geodesic curvature and torsion are discussed in early treatises on curves on surfaces in $\mathbb{R}^3$ (Eisenhart, [9] Lane [16]), but mainly only up to deriving complex formulae for evaluating them in terms of the metric and comparing the geodesic curvature of a curve with the spacial curvature in $\mathbb{R}^3$ and the Gauss curvature of the surface.

More modern books only seem to treat geodesic curvature as a curiousity; as the simplest case when $\nabla_T T \neq 0$, and for Gauss-Bonnet type theorems for p-w continuous curves on surfaces. (Hicks, Willmore, Spivak etc.).

The only other reference for k-lines other than in the surface context is a short paper, with no proofs, by Arnold(14), where again the geometric properties of the lines are not discussed, in particular no use is made of the fact that ∇ is a natural differential given by the geometry of the manifold and there should be no need to have to take the ordinary differentiation usually used when the curves are considered as immersed in some Euclidean space containing the manifold.

Let us consider k-lines with t=arc length so that $\|v(t)\| \equiv 1$ and $v(t) = T(t)$. Hence if X,Y are vectorfields along a k-line c then

$$d/dt \langle X, Y\rangle = \langle \nabla_v X, Y\rangle + \langle \nabla_v Y, X\rangle \quad \text{by Prop}^n\ 0{:}1.$$

Given a geodesic, c, of M consider a vector $E(0) \in T_{c(0)}M$ perpendicular to $v(0)$ and parallel translate it along c; then at each point $c(t)$, $E(t)$ is perpendicular to $v(t)$, and $v(t)$ itself is the parallel translate of $v(0)$, i.e. $\nabla_v v = 0$ and $\nabla_v E = 0$. (fig 1).

Consider a k-line, c, with normals $\{N_1(t), N_2(t), \ldots, N_r(t)\}$. Then as above $\{T(t), N_1(t), \ldots, N_r(t)\}$ remain mutually perpendicular but are no longer the parallel translates of $\{T(0), N_1(0), \ldots, N_r(0)\}$ (fig 2)

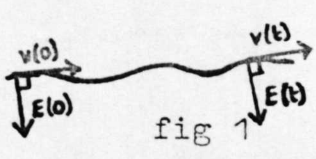


fig 1

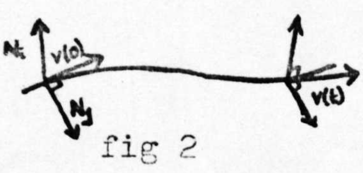
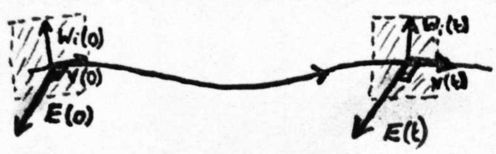


fig 2

However if we take a vector $E(0)$ perpendicular to $T(0),N_1(0),\dots,N_r(0)$, and parallel translating it along c, then by seeing at each point if $E(t)$ remains perpendicular to $T(t),N_1(t),\dots,N_r(t)$ for all t we can say how $(T(t),\dots,N_r(t))$ might be "parallel translated as an $r+1$ frame", though $T(t),\dots,N_r(t)$ are not individually the translates of $T(0),\dots,N_r(0)$.

Definition: An r-frame $(w_1,\dots,w_r)(t)$ along a curve, c, is a parallel translation of an r-frame $(w_1,\dots,w_r)(0)$ at $c(0)$ if given any vector $E(0)$ at $T_{c(0)}M$ perpendicular to $w_i(0)$, $\forall i$, then the parallel translate, $E(t)$, of $E(0)$ remains perpendicular to $w_i(t)$ $\forall i,t$. **In particular this means that E will remain perpendicular to the plane generated by (w,N_i) and we consider this plane at time t to be the parallel translate of the plane at time 0.**



Proposition 1:1 If $c:I\to M$ is a $\underline{k}$-line then for all t $(T(t),N_1(t),\dots,N_r(t))$ is a parallel $(r+1)$-frame.

Proof. $E(t),N_i(t),T(t)$ are vectorfields along c where by definition $\nabla_{v(t)}E=0$.

then using Prop^n 0:1 and the Frenet formulae: A

$$\frac{d}{dt}\langle E(t),v(t)\rangle = \cancel{\langle\nabla_v E,v\rangle} + \langle E,\nabla_v v\rangle = k_1\langle E(t),N_1(t)\rangle$$

$$\frac{d}{dt}\langle E(t),N_1(t)\rangle = \cancel{\langle\nabla_v E,N_1\rangle} + \langle E,\nabla_v N_1\rangle = k_2\langle E,N_2\rangle - k_1\langle E,v\rangle$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d}{dt}\langle E(t),N_i(t)\rangle = k_{i+1}\langle E,N_{i+1}\rangle - k_i\langle E,N_{i-1}\rangle$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d}{dt}\langle E(t),N_r(t)\rangle = -k_r\langle E,N_{r-1}\rangle$$

Setting $f_0 : I \longrightarrow \mathbb{R} : t \longmapsto \langle E(t), v(t) \rangle$

$f_i : I \longrightarrow \mathbb{R} : t \longmapsto \langle E(t), N_i(t) \rangle$ for each i, then

$$\frac{d}{dt}\begin{bmatrix} f_0(t) \\ f_1(t) \\ f_2(t) \\ \cdot \\ \cdot \\ \cdot \\ f_r(t) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & & & & \\ -k_1 & 0 & k_2 & & & \mathbf{0} & \\ 0 & -k_2 & 0 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & 0 & k_{r-1} & 0 \\ & \mathbf{0} & & & -k_{r-1} & 0 & k_r \\ & & & & 0 & -k_r & 0 \end{bmatrix} \cdot \begin{bmatrix} f_0(t) \\ f_1(t) \\ f_2(t) \\ \cdot \\ \cdot \\ \cdot \\ f_r(t) \end{bmatrix}$$

which gives the matrix differential equation:

$$\frac{dX}{dt} = AX \quad \text{s.t. } X(0) = 0 \text{ since } E(0) \perp V(0) \perp N_i(0)$$

Hence $X(t) \equiv 0$ is the unique solution. (Brauer & Nohel [28]),

and $E(t) \perp V(t) \perp N_i(t)$ for all i,t.

q.e.d.

Suppose we add in (n-r-1) vectors $E_1(0), \ldots, E_s(0)$ at $c(0)$ such that $\{v(0), \ldots N_i(0), \ldots E_j(0)\}$ form an n-frame at $c(0)$, and consider the parallel translates: $E_j(t)$ of $E_j(0)$. Putting $F(t) = \{v(t), \ldots, N_i(t), \ldots, E_j(t), \ldots\}$ for all t, then we obtain:

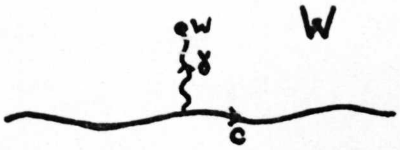
<u>Corollary 1:1</u> $F(t)$ is an orthonormal n-frame at $c(t) \; \forall \, t$.

Proof. By above $v(t) \,\&\, N_i(t) \perp E_j(t)$ for each j

By parallel translation $\langle E_j(t), E_i(t) \rangle = \delta^i_j$

q.e.d.

Hence at each point $c(t)$ of the curve, $F(t)$ gives a basis for $T_{c(t)}M$. These frames can be used to study vectorfields along $\underline{k}$-lines, but also to generalise the idea of Fermi co-ordinates, usually used to study the immediate neighbourhoods of geodesics. (Hicks [13] p133).
i.e. There is an open nhd of the curve, W, and an interval J, such that if $w \in W$ there is a unique geodesic, γ, through w perpendicular to c, at $c(t)$ for some $t \in J$ and $\gamma(0) = c(t)$, $\gamma(1) = w$.



Let $\gamma'(0) = Y_1 N_1(t) + \ldots + Y_{n-1} E_s(t)$ for some t.
Then $w \equiv (t, Y_1, Y_2, \ldots, Y_{n-1})$ gives a local co-ordinate system in a nhd. of c.

We shall use this idea later on, but we can now use it to demonstrate how $\underline{k}$-lines, $\underline{k} \neq 0$, differ from geodesics.

<u>Proposition 1:2</u> A $\underline{k}$-line in M is a geodesic of a local $(n-r)$ hyperplane orthogonal to the geodesic normals, under the induced connexion.

Proof Given a point $c(s)$ of the curve, consider the Fermi co-ordinate nhd about s, and in particular the subset of co-ordinates $W_0 \subset W$ given by $(t, 0, \ldots 0, Y_{r+1}, \ldots Y_{n-1})$ This gives the local $(n-r)$ hyperplane perpendicular to the geodesic normal directions.

We have an induced connexion D on this hyperplane (Hicks [13]) given by $D_X Y = \nabla_X Y - \mathrm{Nor}(X,Y)$

where we have split $\nabla_X Y$ into components tangential, $(D_X Y)$, and normal, $(\mathrm{Nor}(X,Y))$, to v, E_i, and hence to W_0. Taking c as a curve in W_0 then $D_v v = 0$, since $\nabla_v v = k_1 N_1$ perpendicular to v, E_i by construction.

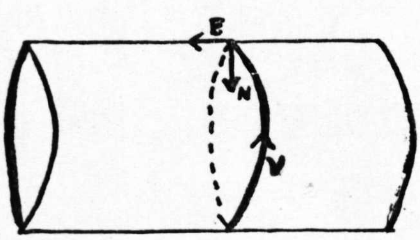
q.e.d.

Intuitively this would suggest that the non-geodesic nature of the curve is restricted to the geodesic normal directions. If $M = \mathbb{R}^n$ and $\mathbf{k} = (k,0,..,0)$ then the above shows that the curve is contained in a 2-plane. We have thus generalised a well known fact about curves in $\mathbb{R}^3$:

<u>Corollary 1:2:1</u> A curve in $\mathbb{R}^n$ with zero torsion is planar.

....

<u>Example</u>: We shall show that a $\underline{k} = (k,0,0)$ line in $\mathbb{R}^3$ is a circle of radius $1/k$. This is a geodesic on the cylinder generated by v and the direction E, perpendicular to v and N.



see p 20.

....

We use F(t) to study vectorfields along c.

<u>Definition</u>. If X is a vectorfield along a $\underline{k}$-line, c, then $X(t) = X^0(t)T(t) + X^1(t)N_1(t) + \ldots + X^{n-1}(t)E_s(t)$. Set

$$X(t) = \begin{bmatrix} X^0(t) \\ X^1(t) \\ \cdot \\ \cdot \\ X^{n-1}(t) \end{bmatrix}$$

The vectorfield can be recovered by premultiplying by

$$[v(t), N_1(t), \ldots, E_s(t)]$$

The $(r+1)$ plane generated by (v,N_i) is parallely translated as a plane, the vectors generating it are not so, but we can gain information about $\underline{k}$-lines by comparing the frame at time t, $\{(v,\ldots,N_r)(t)\}$ say, with the frame formed by parallel translating the vectors of $\{(v,\ldots,N_1)(0)\}$, individually along the $\underline{k}$-line to t.ie we compare the vectors translated as a frame with those translated individually. In particular when do the two frames agree?

If c is our $\underline{k}$-line let $T^p(t)$, $N_j^p(t)$ be the parallel translates of $T(0)$, $N_j(0)$ resp. In general $T^p(t) \neq T(t)$ and $N_j^p(t) \neq N_j(t)$.

By Proposition 1:1

$$T^p(t) = \begin{bmatrix} x_0^0(t) \\ \vdots \\ x_0^r(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots\ldots, \quad N_r^p \equiv \begin{bmatrix} x_r^0(t) \\ \vdots \\ x_r^r(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The non-zero elements of the frame are given by the matrix:

$$Y(t) \equiv \begin{bmatrix} x_0^0(t) & \ldots\ldots & x_r^0(t) \\ \vdots & & \vdots \\ x_0^r(t) & \ldots\ldots & x_r^r(t) \end{bmatrix} \quad \text{st.} Y(0) \equiv I_{r+1}$$

$$\text{Let } A \equiv \begin{bmatrix} 0 & k_1 & 0 & & & \\ -k_1 & 0 & k_2 & & \mathbf{0} & \\ 0 & -k_2 & 0 & \ddots & & \\ & & & \ddots & 0 & k_r \\ & \mathbf{0} & & & -k_r & 0 \end{bmatrix} \quad \text{as before.}$$

Then $Q^2 = -A^2 = \begin{bmatrix} k_1^2 & 0 & -k_1k_2 & & & & \\ 0 & k_1^2+k_2^2 & 0 & & \mathbf{0} & & \\ -k_1k_2 & 0 & k_2^2+k_3^2 & & & & \\ & & & \ddots & & & \\ & \mathbf{0} & & & k_{r-2}^2+k_{r-1}^2 & 0 & -k_rk_{r-1} \\ & & & & 0 & k_{r-1}^2+k_r^2 & 0 \\ & & & & -k_rk_{r-1} & 0 & k_r^2 \end{bmatrix}$

is positive $\begin{cases} \text{definite when } r \text{ is odd.} \\ \text{semi-definite when } r \text{ is even.} \end{cases}$

Also there is an orthogonal P s.t.

$$PQ^2P^{-1} = \begin{bmatrix} \lambda_0^2 & & & & \\ & \lambda_1^2 & & \mathbf{0} & \\ & & \cdot & & \\ & \mathbf{0} & & \cdot & \\ & & & & \lambda_r^2 \end{bmatrix}$$ where $\lambda_i^2 \geq 0$ are the eigenvalues of Q^2.

which justifies the terminology Q^2 above since $Q = P^{-1}\begin{bmatrix} \lambda_0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_r \end{bmatrix}P$

<u>Lemma 1:3</u> $Y(t) = \cos Qt + \mathbf{Q}\sin Qt.T$ for some matrix T.

Proof.

$$\begin{bmatrix} x_0^0(t) \\ \cdot \\ \cdot \\ x_0^r(t) \end{bmatrix} = \begin{bmatrix} \langle T^p(t), T(t)\rangle \\ \cdot \\ \cdot \\ \langle T^p(t), N_r(t)\rangle \end{bmatrix}, \ldots, \begin{bmatrix} x_r^0(t) \\ \cdot \\ \cdot \\ x_r^r(t) \end{bmatrix} = \begin{bmatrix} \langle N_r^p(t), T(t)\rangle \\ \cdot \\ \cdot \\ \langle N_r^p(t), N_r(t)\rangle \end{bmatrix}$$

Using $d/dt\, \langle X,Y\rangle = \langle \nabla_T X, Y\rangle + \langle X, \nabla_T Y\rangle$ and the Frenet formulae:

$$\frac{dY}{dt} = \begin{bmatrix} 0 & k_1 & & & \\ -k_1 & 0 & & \mathbf{0} & \\ & & \cdot & & \\ & & & \cdot & \\ & \mathbf{0} & & 0 & k_r \\ & & & -k_r & 0 \end{bmatrix} \begin{bmatrix} x_0^0 & \cdots & x_r^0 \\ \vdots & & \vdots \\ x_0^r & \cdots & x_r^r \end{bmatrix}$$

i.e $Y'(t) = AY(t)$

$$\begin{aligned}\text{then } Y = \exp At &= I + At + \tfrac{1}{2!}A^2t^2 + \tfrac{1}{3!}A^3t^3 + \ldots \\ &= I + \tfrac{1}{2!}A^2t^2 + \tfrac{1}{4!}A^4t^4 \ldots + At + \tfrac{1}{3!}A^3t^3 + \ldots \\ &= I - \tfrac{1}{2!}Q^2t^2 + \tfrac{1}{4!}Q^4t^4 \ldots + (\qquad)\end{aligned}$$

When r is odd Q^2 is invertible so there is T st. $Q^2T = A$. When r is even, by considering the odd and even columns of Q^2 seperately then each of the columns of A are in the image of Q^2 as a linear operator. So there is T st. $Q^2T = A$, not necessarily unique.

$$\begin{aligned}\therefore Y(t) &= \cos Qt + Q^2Tt - \tfrac{1}{3!}Q^2Q^2Tt + \ldots \\ &= \cos Qt + Q.[Qt - \tfrac{1}{3!}Q^3t^3 + \ldots].T \\ &= \cos Qt + Q\sin QtT.\end{aligned}$$

q.e.d.

For Computation the following is useful:-

<u>Corollay 1:3:1</u>

$$Y(t) = P^{-1}\begin{bmatrix}\cos\lambda_0 t & & 0 \\ & \ddots & \\ 0 & & \cos\lambda_r t\end{bmatrix}P + P^{-1}\begin{bmatrix}\lambda_0\sin\lambda_0 t & & 0 \\ & \ddots & \\ 0 & & \lambda_r\sin\lambda_r t\end{bmatrix}PT.$$

Proof. Use $\sin Qt = \tfrac{1}{2}i(e^{iQt} - e^{-iQt})$, $\cos Qt = \tfrac{1}{2}(e^{iQt} + e^{-iQt})$ and $\exp(P^{-1}AP) = P^{-1}(\exp A)P$

q.e.d.

The above cos, sin form reflects the fact that if we differentiate $Y' = AY$ again then $Y'' = -Q^2Y$.

Proposition 1:5. (i) For $\underline{k} = (k_1,0,..,0)$ then $T(t) = T^p(t), N_1(t) = N_1^p(t)$ for $t = 0, \frac{2\pi n}{k_1}$.

(ii) For $\underline{k} = (k_1,k_2,0,..,0)$ then $N_2(t) = N_2^p(t)$, $T(t) = T^p(t), N_1(t) = N_1^p(t)$ for $t = 0, \frac{2\pi n}{(k_1^2+k_2^2)^{\frac{1}{2}}}$.

(iii) For $r > 2$ then periodicity depends on the eigenvalues of Q^2.

Proof. (i) $A = \begin{bmatrix} 0 & k_1 \\ -k_1 & 0 \end{bmatrix}$ $Q^2 = \begin{bmatrix} k_1^2 & 0 \\ 0 & k_1^2 \end{bmatrix}$ $\lambda_0 = \lambda_1 = k_1$. $P=I$.

Substitution gives

$$Y(t) = \begin{bmatrix} \cos k_1t & 0 \\ 0 & \cos k_1t \end{bmatrix} + \begin{bmatrix} k_1\sin k_1t & 0 \\ 0 & k_1\sin k_1t \end{bmatrix}\begin{bmatrix} 0 & 1/k_1 \\ 1/k_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos k_1t & \sin k_1t \\ -\sin k_1t & \cos k_1t \end{bmatrix} \qquad Y(0) = I,\ Y'(0) = A.$$

(ii) $\underline{k} = (k_1,k_2,0,\ldots,0)$.

$$A = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \qquad Q^2 = \begin{bmatrix} k_1^2 & 0 & -k_1k_2 \\ 0 & k_1^2+k_2^2 & 0 \\ -k_1k_2 & 0 & k_2^2 \end{bmatrix}$$

$$P = \begin{bmatrix} k_2 & 0 & k_1 \\ 0 & 1 & 0 \\ k_1 & 0 & -k_2 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 1/k_1 & 0 \\ \frac{-k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 & \frac{k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}} \\ 0 & 0 & 0 \end{bmatrix}$$

Q^2 has eigenvalues $(k_1^2 + k_2^2)^{\frac{1}{2}}$

$$Y(t) = \begin{bmatrix} \dfrac{k_2^2 + k_1^2\cos(k_1^2+k_2^2)^{1/2}t}{k_1^2+k_2^2} & \dfrac{k_1\sin(k_1^2+k_2^2)^{1/2}t}{(k_1^2+k_2^2)^{1/2}} & \dfrac{k_1k_2(1-\cos(k_1^2+k_2^2)^{1/2}t)}{k_1^2+k_2^2} \\ \dfrac{-k_1\sin(k_1^2+k_2^2)^{1/2}t}{(k_1^2+k_2^2)^{1/2}} & \cos(k_1^2+k_2^2)^{1/2}t & \dfrac{k_2\sin(k_1^2+k_2^2)^{1/2}t}{(k_1^2+k_2^2)^{1/2}} \\ \dfrac{k_1k_2(1-\cos(k_1^2+k_2^2)^{1/2}t)}{k_1^2+k_2^2} & \dfrac{-k_2\sin(k_1^2+k_2^2)^{1/2}t}{(k_1^2+k_2^2)^{1/2}} & \dfrac{k_1^2+k_2^2\cos(k_1^2+k_2^2)^{1/2}t}{k_1^2+k_2^2} \end{bmatrix}$$

Hence $Y(t) = I$ when $t = \dfrac{2\pi n}{(k_1^2+k_2^2)^{\frac{1}{2}}}$.

(iii) Periodicity depends on nature of $\sin Qt$, $\cos Qt$ and hence the eigenvalues of Q^2 i.e. the eigenvalues of $P \sin Qt\, P^{-1}$

eg. If $k_1=k_2=k_3=1$ $k_4=0$ then $\lambda_1 = \lambda_2 = \left(\dfrac{3+5^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}$

$$\lambda_3 = \lambda_4 = \left(\frac{3-5^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}$$

All are thus irrational. Hence $\cos Qt \neq 1$, $\sin Qt \neq 0$ unless $t = 0$; since there is no K st. $\lambda_1 = K\lambda_3$ etc.

q.e.d.

Note: If the curve is not parameterised by arclength then the above shows that periodicity occurs when

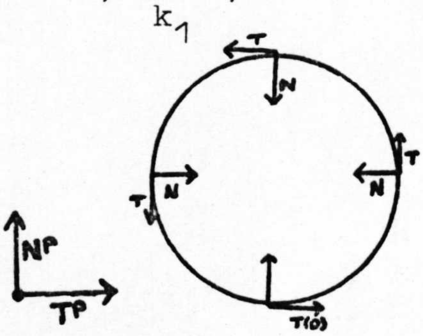
$s = \dfrac{2\pi n}{(k_1^2+k_2^2)^{\frac{1}{2}}}$ ie. $t = \dfrac{2\pi n}{\|v\|(k_1^2+k_2^2)^{\frac{1}{2}}}$

We can use these properties to construct some examples of <u>k</u>-lines. First we extend known examples of <u>k</u>-lines from surfaces in $\mathbb{R}^3$ to <u>k</u>-lines in $\mathbb{R}^n$.

Proposition 1:6 In $\mathbb{R}^n$ then (i) $\underline{k} = (k_1, 0, .., 0)$ lines are circles of radius $1/k_1$. and (ii) $\underline{k} = (k_1, k_2, 0, .., 0)$ lines are cylindrical helices of radius $\dfrac{k_1}{(k_1^2 + k_2^2)^{\frac{1}{2}}}$ and pitch $\dfrac{2\pi k_2}{(k_1^2 + k_2^2)^{\frac{1}{2}}}$.

Proof (i) Proposition 1:1 says that in $\mathbb{R}^n$ $(T(t), N_1(t))$ is a parallel 2-frame, that is planar.

Proposition 1:5 says that $T(t) = T^p(t)$, $N_1(t) = N_1^p(t)$ for $t = 0, \dfrac{2\pi n}{k_1}$, $n \in \mathbb{Z}$.



$$T(t) = T^p \cos k_1 t + N_1^p \sin k_1 t$$
$$N(t) = -T^p \sin k_1 t + N_1^p \cos k_1 t$$

Since (T^p, N_1^p) are the (x,y) axes translated to $c(t)$ the curves must be circles.

Let r be the radius of the $\underline{k}$-line. Circumference $= 2\pi r$ and velocity $\equiv 1$. Hence $\dfrac{2\pi r}{2\pi/k_1} = 1$ ie. $r = 1/k_1$.

So $(k_1, 0, \ldots, 0)$ lines in $\mathbb{R}^n$ are circles of radius $1/k_1$ in direct analogy of magnetic fields. Note also that when $k_1 = 0$ then we obtain straight lines, the geodesics of $\mathbb{R}^n$.

(ii) $\underline{k} = (k_1, k_2, 0, .., 0)$

By calculation $\Upsilon^{-1}(t) = \begin{bmatrix} \dfrac{k_2^2 + k_1^2 \cos\lambda t}{\lambda^4} & \dfrac{-k_1 \sin\lambda t}{\lambda^3} & \dfrac{k_1 k_2 (1 - \cos\lambda t)}{\lambda^4} \\ \dfrac{k_1 \sin\lambda t}{\lambda^3} & \cos\lambda t & \dfrac{-k_2 \sin\lambda t}{\lambda^3} \\ \dfrac{k_1 k_2 (1 - \cos\lambda t)}{\lambda^4} & \dfrac{k_2 \sin\lambda t}{\lambda^3} & \dfrac{k_1^2 + k_2^2 \cos\lambda t}{\lambda^4} \end{bmatrix}$

$\lambda = (k_1^2 + k_2^2)^{\frac{1}{2}}$

ie. If $c: I \longrightarrow M$ is a $\underline{k}$-line then

$$c'(t) = \left[\frac{k_2^1 + k_2^2\cos\lambda t}{\lambda^4}\right] T^p + \left[\frac{k_1\sin\lambda t}{\lambda^3}\right] N_1^p + \left[\frac{k_1k_2(1-\cos\lambda t)}{\lambda^4}\right] N_2^p$$

Consider the rotation of co-ordinates given by

$$\begin{bmatrix} 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \text{where } \cos\theta = k_1/\lambda, \quad \sin\theta = -k_2/\lambda$$

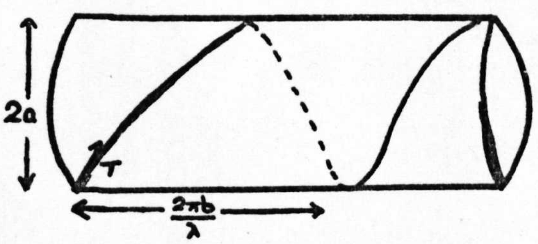
then under this transformation

$$c'(t) = \left[\frac{-k_1\sin\lambda t}{\lambda^3}, \frac{k_1\cos\lambda t}{\lambda^3}, \frac{k_2}{\lambda^3}\right]$$

$$c(t) = \left[\frac{k_1\cos\lambda t}{\lambda^2}, \frac{k_1\sin\lambda t}{\lambda^2}, \frac{k_2 t}{\lambda^3}\right]$$

Put $a = \dfrac{k_1}{\lambda^2}$, $b = \dfrac{k_2}{\lambda^3}$ then $c(t) = [a\cos\lambda t, a\sin\lambda t, bt]$

which is the equation of a cylindrical helix of radius a, and pitch b.



When $k_2 = 0$ then we have circles radius $1/k_1$.

As a corollary we can obtain the $\underline{k}$-lines on S^n for $\underline{k} = (k_1, 0, \ldots, 0)$. We need the following from Hicks ([13])

<u>Definition</u> Let c be a curve in S^n with $c'(t) = T(t)$ then let $V(T,T) = (D_T T - \nabla_T T)$ where ∇ is the induced connexion on S^n from the Euclidean connexion D on $\mathbb{R}^{n+1}$. Put $k_T = \|V(T,T)\|$

Corollary 1:6:1 The $\underline{k}$-lines on S^n are circles in $\mathbb{R}^{n+1}$ of radius $1/(1+k_1^2)^{\frac{1}{2}}$ for $\underline{k} = (k_1, 0, \ldots 0)$.

Proof. The geodesics on S^n are the great circles of radius 1. We need the following theorem

(Meusnier)(Hicks p 77) If $\underline{M} \subset M$ is a submanifold of M with connexion ∇ induced from that D on M then $\overline{k}_1^2 = k_1^2 + k_T^2$ relates the geodesic curvature of a curve c in $\underline{M}$, $\underline{k}_1$, to the geodesic curvature of c in M, k_1. k_T is the normal curvature of $\underline{M}$ in M.

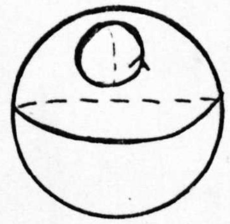
To calculate k_T for $S^n \subset \mathbb{R}^{n+1}$ consider a geodesic of S^n, so $\nabla_T T = 0$. As a curve in $\mathbb{R}^{n+1}$ this is a circle of radius 1, and so by the previous proposition is a $\underline{k} = (1,0,\ldots,0)$ line of $\mathbb{R}^{n+1}$. Hence $D_T T = 1.N_1$ and so $k_T = \|1.N_1\| = 1$.

Hence given a $\underline{k} = (k,0,\ldots,0)$ line on S^n, Meusnier's theorem says that the curve is a line in $\mathbb{R}^{n+1}$ of geodesic curvature $k_g^2 = 1 + k^2$.

Applying Proposition 1:6 again then this is a circle in $\mathbb{R}^{n+1}$ of radius $1/(1+k^2)^{\frac{1}{2}}$.

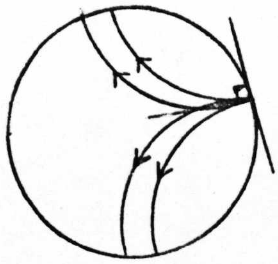
q.e.d.

Example. On S^2.

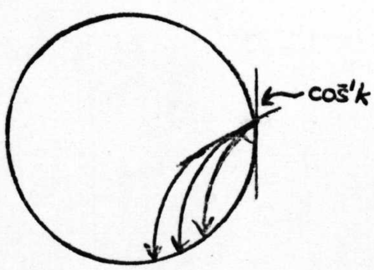


Note: When $k = 0$ then we get circles of radius 1, the great circles.

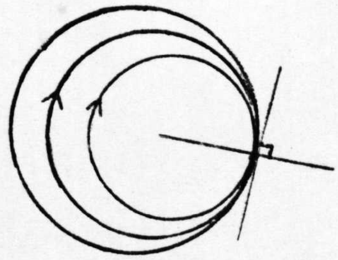
For completeness we catalogue Arnold's ([4]) example of $\underline{k}$-lines on a manifold of negative curvature, the Lobachevsky plane, which we shall use extensively as an example. The manifold is the inside of the unit circle with metric $ds^2 = \dfrac{dx^2 + dy^2}{(1-x^2-y^2)}$; a manifold of constant negative curvature -1.



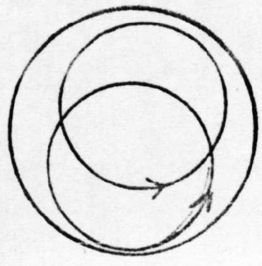
$\underline{k} = 0$ geodesic



$0 \leq k < 1$ hypercycle



$k = 1$ horocycle



$k \geq 1$ epicycle.

Section 2: k-flows on manifolds.

One of the best documented of flows on a manifold is the geodesic flow on the unit tangent bundle of a Riemann manifold. (eg Anosov [2]). The definition depends on the fact that through any point on the manifold, then given any initial velocity there is a unique geodesic line in the manifold with these initial conditions. This yields a flow on T^1M by taking as the flow line through (x,v) the lift of the geodesic with initial conditions (x,v).

i.e. $$\varphi_t : T^1M \longrightarrow T^1M$$
$$:(x,v) \longmapsto (x(t),v(t))$$

where $x(t)$ is the unique geodesic through x with $\dot{x}(t) = v(t)$ and $\dot{x}(0) = v$.

The flow is generated by a second order equation on M, namely $Z : T^1M \longrightarrow TT^1M$

$$(x,v) \longmapsto (x,v,v \oplus 0)$$

using covariant co-ordinates. (Appendix A).

We should like to describe more general flows on T^1M based not on the geodesics of M, but on lines of known geodesic curvature :- $\underline{k}$-lines. This is not so easy as the geodesic case since we lose the property that for each point and each initial direction we have a unique line in M; if we look at the Frenet vectors then there is a unique $\underline{k}$-line at a point $m \in M$ iff we specify an initial $(r+1)$ frame, giving a whole family of $\underline{k}$-lines for each initial direction $v \in T^1_mM$.

We get round this problem by specifying for each $v \in T^1M$ an $(r+1)$ frame, $(x,v,N_1,\ldots,N_r) \in F^{r+1}_mM$, by means of a section $N_{\underline{k}} : T^1M \longrightarrow F^{r+1}M$. We have to impose some conditions on this section, for if a $\underline{k}$-line of the flow has geodesic normals at t=0, given by $N_{\underline{k}}(x,v)$ then at a later time, t, the normals ~~must~~ still be given by $N_{\underline{k}}(x(t),v(t))$.

There is obviously a unique $\underline{k}$-flow on $F^{r+1}M$ for each $\underline{k}$, so why try to study a restricted flow on T^1M? To answer this we approach the problem from the point of view of mechanics. If we consider a Lagrangian, L, of the form $L = T - V$, (Abraham [1], or Appendix B), where T is the kinetic energy and V, a potential energy term depending only on position. This corresponds physically to the motions of particles on a space under no external forces. If we take a more general case and include in the Lagrangian a term, A, quadratic in the velocity, $L = T - V - A$, then we generate a $\underline{k}$-flow, and physically we are now studying the motions of particles under forces EXTERNAL to the manifold. (magnetic fields etc). Asking which sections $N_{\underline{k}}$ generate $\underline{k}$-flows on T^1M then becomes the problem of finding the analogues of Flemings' right hand rules for establishing in which direction the lines will bend away from the straight.

Finally it is known that the $\underline{k}$-flow on $F^{r+1}M$ and the geodesic flow on T^1M preserve the natural Riemann volume element. (Appendix A). In looking at this problem for $\underline{k}$-flows on T^1M we come across the term $\mathrm{Trace}(DN_1) - \langle DN_1(v).v,v\rangle$ which arises in the study of Brownian motion on a manifold, but with a 'drift' term normal to the direction of motion. (Elworthy [11]).

Definitions

(a) A $\underline{k}$-flow on T^1M is a flow on T^1M, whose flow lines are the canonical lifts of $\underline{k}$-lines of M.

(b) Given a $\underline{k}$-flow on T^1M then the $\underline{k}$-section of the flow is the map $N_{\underline{k}} : T^1M \to F^{r+1}M : (x,v) \mapsto (x,v,N_1,\ldots,N_r)$, where (N_i) are the geodesic normals of the flow line through (x,v).

(c) Given any smooth section $\zeta : T^1M \to F^{r+1}M : (x,v) \mapsto (x,v,\zeta_1,\ldots,\zeta_r)$ and any $\underline{k}$ let $Z_{\underline{k}} : T^1M \to TT^1M : (x,v) \to (x,v,v \oplus k_1\zeta_1)$.

Lemma 2:1 Given a section $\zeta : T^1M \to F^{r+1}M$ and any $\underline{k}$ then $Z_{\underline{k}}$ is well defined.

Proof. Given charts (U,φ) and (V,ψ) st. $U \cap V \neq \emptyset$ and $g = \psi^{-1}\circ\varphi$ the chart transform then

$$\begin{array}{ccc} T^1U & \xrightarrow{\zeta} & F^{r+1}U \\ \downarrow{\scriptstyle Tg} & \circlearrowleft & \downarrow{\scriptstyle F^{r+1}_g} \\ T^1V & \xrightarrow{\zeta} & F^{r+1}V \end{array} \qquad \begin{array}{ccc} (x,v) & \longmapsto & (x,v,\zeta_1,\ldots,\zeta_r) \\ \downarrow & & \downarrow \\ (g(x),Dg(x)v) & \longmapsto & (g(x),Dg(x)v,\ldots,Dg(x)\zeta_r) \end{array}$$

and this implies:-

$$\begin{array}{ccc} T^1U & \xrightarrow{Z_{\underline{k}}} & TT^1U \\ {\scriptstyle Tg}\downarrow & \circlearrowleft & \downarrow{\scriptstyle T^2g} \\ T^1V & \xrightarrow[Z_{\underline{k}}]{} & TT^1V \end{array} \qquad \begin{array}{ccc} (x,v) & \longmapsto & (x,v,v \oplus k_1\zeta_1) \\ \downarrow & & \downarrow \\ (g(x),Dg(x)v) & \longmapsto & (g(x),Dg(x),Dg(x)v \oplus k_1Dg(x)\zeta_1). \end{array}$$

q.e.d.

Lemma 2:2 If $N_{\underline{k}}$ is the $\underline{k}$-section of a $\underline{k}$-flow then $Z_{\underline{k}}$ is the velocity vectorfield of the flow.

Proof. Consider a chart (U,φ) of M. By definition the velocity vectorfield $X : T^1U \to TT^1U : (x,v) \mapsto (x,v,\dot{x} \oplus \dot{v} + \Gamma(x)(v,v))$

Since each flow line is the canonical lift of a $\underline{k}$-line of M then $\dot{x} = v$ and $\dot{v} + \Gamma(x)(v,v) = \nabla_v v = k_1N_1$.

So $X \equiv Z_{\underline{k}}$.

Notation: Let $X_{\underline{k}} : F^{r+1}M \longrightarrow TF^{r+1}M : (x,v,\xi_1,\dots, \xi_r) \longmapsto$
$(x,v,\dots,\xi_r)(v \oplus k_1\xi_1 \oplus \dots \oplus k_{i+1}\xi_{i+1} - k_i\xi_{i-1} \oplus \dots \oplus -k_r\xi_{r-1})$
generate the $\underline{k}$-flow on $F^{r+1}M$.

<u>Lemma 2:3</u> If $N_{\underline{k}}$ is the $\underline{k}$-section of a $\underline{k}$-flow then
$TN_{\underline{k}} \circ Z_{\underline{k}} = X_{\underline{k}} \circ N_{\underline{k}}$.

Proof.

$$X_{\underline{k}} \circ N_{\underline{k}} : T^1M \longrightarrow F^{r+1}M \longrightarrow TF^{r+1}M.$$
$$(x,v) \longmapsto (x,v,N_1,\dots,N_r) \longmapsto (x,v,\dots,N_r)(v \oplus \dots \oplus -k_r N_{r-1})$$

$$N_{\underline{k}} : T^1M \longrightarrow F^{r+1}M : (x,v) \longmapsto (x,v,N_1(x,v),\dots,N_r(x,v))$$

So

$TN_{\underline{k}}(x,v,e_1 \oplus e_2 + \Gamma(x)(v,e_1)) =$
$(x,\dots,N_r)(v \oplus k_1N_1 \oplus DN_1(x,v)(v,e_2) + \Gamma(N_1,v) \oplus \dots \oplus DN_r(x,v)(v,e_2) +$
$\Gamma(x)(N_r,v))$

where $e_2 = k_1N_1 - \Gamma(x)(v,v)$.

However $N_{\underline{k}}$ is a $\underline{k}$-section, and so:-
$\dot{x} = v, \nabla_v v = k_1N_1, \dots, \nabla_v N_r = -k_rN_{r-1}$; since (N_i) are the geodesic normals of the $\underline{k}$-lines through (x,v).

Hence we obtain:-
$\dot{v} = \nabla_v v - \Gamma(x)(v,v) = -\Gamma(x)(v,v) + k_1N_1 = e_2$.
and by the chain rule:-
$\dot{N}_1 = DN_1(x,v)(\dot{x},\dot{v}) = DN_1(x,v)(v,e_2) = -\Gamma(x)(v,N_1) + k_2N_2 - k_1v).$
$\vdots$
$\dot{N}_r = DN_r(x,v)(\dot{x},\dot{v}) = DN_r(x,v)(v,e_2) = -\Gamma(x)(v,N_r) - k_rN_{r-1}.$

$\therefore TN_{\underline{k}} \circ Z_{\underline{k}}(x,v) =$
$(x,\dots,N_r)(v \oplus k_1N_1 \oplus \cancel{-\Gamma(x)(v,N_1)} + k_2N_2 - k_1v + \cancel{\Gamma(x,v)(v,N_1)}) \oplus \dots$

$$\ldots\ldots \oplus -\cancel{\Gamma(x,)(v,N_r)} - k_r N_{r-1} + \cancel{\Gamma(x)(v,N_r)}).$$
$$= X_{\underline{k}} \circ N_{\underline{k}}(x,v).$$

q.e.d.

<u>Lemma 2:4</u>. Given a section $\mathcal{J} : T^1M \longrightarrow F^{r+1}M$ then if $T\mathcal{J} \circ Z_{\underline{k}} = X_{\underline{k}} \circ \mathcal{J}$ for some $\underline{k}$, then $\mathcal{J}$ is the $\underline{k}$ section of a $\underline{k}$-flow.

Proof. Let $\mathcal{J} : T^1M \longrightarrow F^{r+1}M : (x,v) \longmapsto (x,v,\mathcal{J}_1,\ldots\ldots,\mathcal{J}_r)$

If $\underline{k} = (k_1,\ldots,k_r,0,\ldots,0)$ then

$Z_{\underline{k}} : T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus k_1\mathcal{J}_1)$ is a well defined vectorfield on T^1M by Lemma 2:1. Let $c : I \to T^1M$ be an integral curve of $Z_{\underline{k}}$. Then $(\mathcal{J} \circ c) : I \longrightarrow F^{r+1}M$.

$$d/dt\,(\mathcal{J} \circ c)(t) = T(\mathcal{J} \circ c)(t,1) = T\mathcal{J} \circ Tc(t,1) = T\mathcal{J} \circ c'(t) =$$
$$T\mathcal{J} \circ Z(c(t)) = X_{\underline{k}} \circ \mathcal{J}(c(t)). \text{ by hypothesis.}$$

Hence $(\mathcal{J} \circ c)$ is a solution curve of $X_{\underline{k}}$, and if $p : F^{r+1}M \longrightarrow M$ is the projection map then $(p \circ \mathcal{J} \circ c)$ is a $\underline{k}$-line of M, with geodesic normals $(\mathcal{J}_1,\ldots,\mathcal{J}_r)$, by definition of the $\underline{k}$-flow on $F^{r+1}M$.

Since
$$\begin{tikzcd} T^1M \arrow[rr, "\mathcal{J}"] \arrow[dr, "\pi_1"'] & G & F^{r+1}M \arrow[dl, "p"] \\ & M & \end{tikzcd}$$
then $(\pi_1 \circ c) = p \circ \mathcal{J} \circ c$.

$\therefore$ $(\pi_1 \circ c)$ is a $\underline{k}$-line of M with normals $(\mathcal{J}_1,\ldots,\mathcal{J}_r)$ and $Z_{\underline{k}}$ generates a $\underline{k}$-flow, with $\mathcal{J}$ as $\underline{k}$-section.

q.e.d.

These Lemmas allow us to prove:-

<u>Proposition 2:5</u>. An arbitrary section $N : T^1M \longrightarrow F^{r+1}M$ is the $\underline{k}$-section of a $\underline{k}$-flow iff $TN \circ Z_{\underline{k}} = X_{\underline{k}} \circ N$.

Examples. (1) In $\mathbb{R}^{2n}$, with co-ords $(x_1,\dots,x_{2n})$ let

$$N(x^i, v^i) = \left[x^i, \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{2n} \end{bmatrix} \right]$$

By the computation in Lemma 2:3 it is sufficient to show:- $DN(x,v)(-\Gamma(x)(v,v) + k_1N_1(v)) = -k_1v + \Gamma(x)(v,N_1)$

i.e. $DN_1 \circ N_1(x,v) = -v$ since $\Gamma \equiv 0$.

$$DN_1(x,v)(x, DN_1(x,v)) = \left(x, \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_{2n} \end{bmatrix}\right) = (x,-v).$$

(2) On the Lobachevsky plane if $(x,v) = (x_1,x_2,v_1,v_2)$

let $N(x,v) = (x_1,x_2,v_2,-v_1) = \left(x_1,x_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$.

That is we have chosen those normals forming a right hand system with v. By the above $DN(x,v).N(v) = -v$. Hence it is sufficient to prove that $DN(\Gamma(x)(v,v)) = \Gamma(x)(v,N)$.

Now $g_{ij} = \delta^i_j / A^2$ where $A^2 = (1-x^2-y^2)$.

By computation using $\Gamma^i_{jk} = \frac{1}{2}g^{ip}(g_{jp.k} + g_{kp.j} - g_{jk.p})$

$$\Gamma^1_{ij} = \frac{1}{2A}\begin{bmatrix} x & y \\ y & -x \end{bmatrix}$$

$$\Gamma^2_{ij} = \frac{1}{2A}\begin{bmatrix} -y & x \\ x & y \end{bmatrix}$$

Hence $\Gamma(x)(v,v) = \frac{1}{2A}\left\{(xv_1^2 + 2yv_1v_2 - xv_2^2), (-yv_1^2 + 2xv_1v_2 + yv_2^2)\right\}$

$\therefore DN.\Gamma(v,v) = \frac{1}{2A}\left\{(yv_2^2 + 2xv_1v_2 - yv_1^2), (xv_2^2 - 2yv_1v_2 - xv_1^2)\right\}$

$= \Gamma(x)(v,N)$.

(3) Any compact surface of genus >1 has the Lobachevsky plane, L, as a universal covering space with fundamental group, G, isomorphic to a Fuchsian group.(a discrete subgroup of $LF(2,\mathbb{R})$). Hence the surface can be represented as L/G. With the metric above, picking left or right hand normals determines a k- flow for given k, on L.

Elements of Fuchsian groups act as <u>isometries</u> on L, so factoring L by G to obtain the surface induces a $\underline{k}$- flow on the surface. (see Hopf, McBeath).

We should like to know what happens when we 'reverse' the flow so we introduce the following ideas:

Lemma 2:6. If $c : I \to M$ is a $\underline{k}$-line with geodesic normals $(N_1(v), \ldots, N_r(v))$ then $-c : I \to M : t \mapsto c(-t)$ is a $\underline{k}$-line with normals $N'_{\underline{k}} = (\ldots, (-1)^{i+1} N_i(-v), \ldots)$.

Proof: If $c : I \to M$ is a $\underline{k}$-line then $\dot{c}(t) = v$, $\nabla_v v = k_1 N_1(v)$, $\nabla_v N_1 = k_2 N_2 - k_1 v, \ldots, \nabla_v N_i = k_{i+1} N_{i+1} - k_i N_{i-1}, \ldots,$ $\nabla_v N_r = -k_r N_{r-1}$. by Frenet.

Hence $-c : I \to M$ has velocity $-\dot{c}(t) = -v(t) = w(t)$ say

$$\nabla_w w = \nabla_{-v}(-v) = \nabla_v v = k_1 N_1(v) = k_1 N_1(-w) = k_1 N'_1(w) \text{ say.}$$

$$\nabla_w N'_1(w) = -\nabla_v N_1(v) = k_2(-N_2(v)) - k_1(-v) = k_2 N'_2(w) - k_1 w \text{ say,}$$

$$\nabla_w N'_2(w) = \nabla_v N_2 = k_3 N_3(v) - k_2 N_1(v) = k_3 N'_3(w) - k_2 N'_1(w) \text{ etc}$$

Setting $N'_i(w) = (-1)^{i+1} N_i(-w)$ then

$$\nabla_w N_1(w) = (-1)^i \nabla_v N_i(v) = k_{i+1}(-1)^{i+2} N_{i+1} - k_i(-1)^i N_{i-1}$$

$$= k_{i+1} N'_{i+1}(w) - k_i N'_{i-1}(w)$$

So $-c$ is a $\underline{k}$-line with normals N'_i as above.

q.e.d.

Lemma 2:7. If $N_{\underline{k}}$ is the $\underline{k}$-section of a $\underline{k}$-flow then $N'_{\underline{k}}$ is the $\underline{k}$-section of a $\underline{k}$-flow whose $\underline{k}$-lines are the $\underline{k}$-lines of the first flow with opposite orientation.

Proof: $Z_{\underline{k}} : T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus k_1N_1(v))$
generates the $\underline{k}$-flow.

$$Z_{\underline{k}} : T^1M \longrightarrow TT^1M : (x,w) \longmapsto (x,w,w \oplus k_1N_1(-w))$$

generates a flow. By definition of vectorfield $\nabla_w w = k_1N_1(-w)$ and by the work in the previous lemma $\nabla_w N_i'(w) = k_{i+1}N_{i+1}' - k_iN_{i-1}'$ etc. Hence we have a $\underline{k}$-flow with $\underline{k}$-section $N_{\underline{k}}'$.

q.e.d.

Definition: A $\underline{k}$-flow is reversible if $N_{\underline{k}} = N_{\underline{k}}'$.
This implies that if c is a $\underline{k}$-line of the flow then -c is also a $\underline{k}$-line of the $\underline{k}$-flow.

Examples: (i) Geodesic flow.

(ii) A $\underline{k}$-flow on the Lobachevsky plane is NOT reversible, since if c is a $\underline{k}$-line of the $\underline{k}$-flow for which (v,N_1) forms a right hand frame then -c is a $\underline{k}$-line with normals N_1' st (v,N_1') form a left hand system.

This opens the question of the different types of flow, for instance the $\underline{k}$-flow on an oriented surface is not reversible but it is 'returnable' in the sense that if two points of the manifold are connected by a $\underline{k}$-line then in the reverse direction we can connect them by a $\underline{k}$-line of the same flow not necessarily the reverse of the original $\underline{k}$-line. This happens in the example above

We do not pursue these ideas in this thesis but we shall later need:-

Notation: If $N_{\underline{k}}$ is the $\underline{k}$-section of a $\underline{k}$-flow let $N'_{\underline{k}}$ be the $\underline{k}$-section of the reverse flow, as defined above.

Volume preserving properties.

It is known that the geodesic flow on T^1M preserves the Riemann volume element (Sasaki [23]), and in Appendix A we show that the $\underline{k}$-flows on $F^{r+1}M$ are volume preserving.

To study the $\underline{k}$-flows on T^1M we use a lemma by Sasaki which will use the property that the flow on T^1M is the restriction of a flow on TM,

Lemma 2:8. A $\underline{k}$-flow on T^1M is the restriction of a $\underline{k}$-flow on TM.

Proof. Recall that the flow line of the $\underline{k}$-flow on T^1M through $v \in T^1_mM$ is the lift of a $\underline{k}$-line in M through $m \in M$ with initial velocity v. So given $w \in T_mM$ we get a flow line by taking the $\underline{k}$-line of the flow on T^1M through $v = w/\|w\|$. Since a $\underline{k}$-line has constant speed at all points ($v/\|v\| = T, \nabla_T T = k_1 N_1$ etc), we can reparameterise the curve to have speed $\|w\|$, then take the canonical lift into TM to obtain a $\underline{k}$ - line flow. If $\|w\| = 0$ we take the stationary flow line as usual.

To show that this collection of lines actually generates a flow on T M consider the maps:

$$Z_{\underline{k}} : T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus k_1N_1(x,v))$$

and the extension of N_1:

$$N : TM \longrightarrow TM : (x,v) \longmapsto \begin{cases} \|v\|^2 N_1(x, v/\|v\|) & v \neq 0 \\ 0 & v = 0 \end{cases}$$

Then the vectorfield

$$Z : TM \longrightarrow TTM : (x,v) \longmapsto (x,v,v \oplus k_1N(x,v))$$

generates the extension, since the above implies that $\dot{x} = v$ and $\nabla_V v = k_1 N(x,v)$. ie. because $\|v\| =$ constant $\nabla_T T = k_1 N_1(x,v)$.

q.e.d.

We can now use a lemma of Sasaki([23] ,p151): If $N^{n-1} \subset M^n$ is a Riemann submanifold of the Riemann manifold, M, with unit inclusion normal, e, and Z is a vectorfield on M st Z/N is a vectorfield on N, then Z preserves the Riemann volume element, Ω_N, of N iff $\mathrm{div}_{\Omega_N} Z = Z^A{}_{,A} - Z_{B,C}e^B e^C = 0$, where (x^A) is a coordinate system on M, div is divergence and (,) denotes covariant derivation.

We can now apply this to $T^1M \subset TM$ and the extended flow generated above. T^1M is contained in TM as a submanifold of constant Kinetic energy under the map $K : TM \to \mathbb{R} : (x,v) \mapsto \frac{1}{2} g(x)(v,v)$, being invariant under the flow since $\|v\| \equiv$ constant.

<u>Definition</u>: Let $(x^i, v^j \,;\, i,j=1,..,n)$ be a co-ordinate chart on TM. Then for the map $N : TM \to TM$ let $D_V N = [\partial N^i/\partial v^j]$ $i,j=1,..,n$ i.e. part of the usual Jacobian associated with TN.

<u>Lemma2 :9</u>. A <u>k</u>-flow on T^1M preserves the natural volume element of T^1M iff $\mathrm{trace}.D_V N \equiv 0$.

Proof. Using the standard, not covariant, co-ordinates on $(TT^1U, TT^1\varphi)$, there is a unit normal

$$e : (x^1,\dots,x^n,v^1,\dots,v^n) \to (x^1,\dots,v^n)(0,\dots,0,v^1,\dots,v^n)$$

ie. if we write $(\partial/\partial x^A) = (\partial/\partial x^1,\dots,\partial/\partial x^n, \partial/\partial v^1,\dots, \partial/\partial v^n)$, $A = 1,\dots,2n$ as the induced basis for TT^1U, then $(e^A) = (0,\dots,0,v^1,\dots,v^n)$.

$$Z : (x,v) \longrightarrow (x,v,v,-\Gamma(x)(v,v)+k_1N(x,v)).$$

$$\therefore (Z^A) = (v^1,\dots,v^n, -\Gamma^1_{ij}v^iv^j,\dots, -\Gamma^n_{ij}v^iv^j)$$

By Sasaki [23] p345 or Appendix A

$$Z^A_{,A} = \frac{\partial Z^i}{\partial x^i} + \frac{\partial Z^{n+i}}{\partial v^i} + \tfrac{1}{2}G^{JK}\cdot\frac{\partial G_{JK}}{\partial x^I}$$

$$= 0 + \frac{\partial}{\partial v^i}(-\Gamma^i_{jk}v^jv^k + k_1N(x,v)) + 2\Gamma^h_{hi}v^i$$

$$= k_1\sum_i \frac{\partial N^i(x,v)}{\partial v^i} = k_1 \text{ trace}.D_vN.$$

where G is the induced Riemann metric on TM (App A).

$$Z_A = G_{AB}Z^B \quad \therefore Z_{n+j} = G_{n+j,k}Z^k + G_{n+j,n+k}Z^{n+k}$$

$$\therefore Z_{n+j} = [pk,j]\,v^kv^p + g_{jk}(-\Gamma^k_{pq}v^pv^q + k_1N^i)$$
$$= [pk,j]\,v^kv^p + k_1g_{jk}N^k - [pq,j]\,v^pv^q.$$
$$= k_1g_{jk}N^k(v).$$

To calculate $Z_{B,C}\,e^Be^C$ note that $e^i=0$ $i=1,..,n$.

So $Z_{B,C}\,e^Be^C = Z_{n+j,n+k}\,v^jv^k$

$$= \left(\frac{\partial Z_{n+j}}{\partial v^k} - \Gamma^A_{n+j\ n+k}\right)v^jv^k$$

$$= \frac{\partial Z_{n+j}}{\partial v^k}v^jv^k \qquad \text{since } \Gamma^A_{n+j\,n+k}\text{, the Christoffel symbols for } G, = 0$$

$$= k_1\left(\frac{\partial}{\partial v^k}(g_{jp}(x)N^p(x,v)\right)v^jv^k$$

$$= k_1g_{jp}(x)\frac{\partial N^p}{\partial v^k}v^jv^k = \langle D_vN.v, v\rangle_x$$

Note that at this point we have:

$$\text{div}_N Z = \text{trace}D_vN - \langle D_vN.v, v\rangle$$

$$\text{Now } \langle D_v N.v\,,\,v\rangle = \langle \lim_{h\to 0} 1/h\,(N(v+hv)-N(v)),\,v\rangle$$
$$= \langle \lim_{h\to 0} 1/h\,(\|v+hv\|^2 - \|v\|^2)N_1(v/\|v\|),\,v\rangle$$
$$= 0$$

q.e.d.

It is felt that in view of a result in Abraham (1) which states that given two volumes on an orientable manifold N, Ω_1 and Ω_2 say, then given a vectorfield X on N

$$\mathrm{div}_{\Omega_2} X = \mathrm{div}_{\Omega_1} X + \frac{L_X f}{f}$$

where L is the Lie derivative and $f : N \longrightarrow \mathbb{R}$ st $f(m) \neq 0$ for $m \in N$. Now L_X is the derivative of f along the flow lines of X, so if we ask the question "Is there a volume on T^1M with respect to which the k-flow is incompressible?" then we have to solve for a function, f, st $L_{Z_{\underline{k}}} f = f \,\mathrm{div}\, Z_{\underline{k}}$. Since we know $\mathrm{div}_{\Omega} Z_{\underline{k}} = \mathrm{tr} DN(v)$ by restricting attention to flow lines we should be able to reverse the process and integrate along the flow lines to find such an f.

<u>Examples of volume preserving k-flows:</u>

(i) geodesic flow, since $N = 0$.

(ii) $\underline{k}$- flows on the Lobachevsky plane, since $N \equiv \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix}$

(iii) $\underline{k}$-flows on 2n-manifolds with N_1 given by:

$$N_1(x,v) = \left(x\,, \begin{bmatrix} 0 & \pm I_n \\ \mp I_n & 0 \end{bmatrix}\begin{bmatrix} \underline{v} \\ \ \end{bmatrix}\right).$$

Section 3: Variation Theory for k-flows.

In these next two sections we develop a method for studying the orbits of k-flows, or more general flows on T^1M. In particular we study the flowlines close to a given orbit, as usual in the Calculus of Variations.

Given a flow $\varphi_t : T^1M \to T^1M$ then $T\varphi_t : TT^1M \to TT^1M$ is a 'linear approximation' to the flow, and we show the connection with the orbits of the flow in T^1M.

Definition: A flow φ_t on T^1M is a unit tangent flow if it is generated by a vectorfield $Z : T^1M \to TT^1M$ such that $T\pi_1 \circ Z = id$.

Example : Given a 2nd order equation on M,(a vf $Z: TM \to TTM$ st $T\pi_1 oZ = id$) st T^1M is a flow invariant subspace of TM then Z/T^1M is a unit tangent flow.eg (i) T^1M is a constant energy surface for the geodesic flow.(KE $\equiv$ 1, see appendix B) (ii) a k-flow.

Definition: Given a flow line $c: I \to T^1M$ of a unit tangent flow then the curve $(\pi_1 \circ c) : I \to M$ is the corresponding sub-flow line. For a k-flow a sub-flow line is a k-line of M.

Lemma 3:1. A flow line of a unit tangent flow is the canonical lift of its sub-flow line.

Proof: As for second order equations if $d : I \to M$ is a sub-flow line then $d = \pi_1 oc$ for some curve c in T^1M. d has a velocity $\dot{d}(t) \equiv Td(t,1)$, this being the canonical lift into T^1M, since $\dot{d}: I \to TM$.

Now $Td(t,1) = T(\pi_1 o c)(t,1) = T\pi_1 o Tc(t,1)$

$= T\pi_1(\dot{c}(t)) = T\pi_1(Z(c(t)))$ by def^n of c

$=(T\pi_1 o Z)(c(t))$

$= c(t)$ by def^n of unit tangent flow.

q.e.d

This allows us to study the flow lines of a unit tangent flow via their projections into M, where the differential geometry of M itself can be used.

Lemma 3:2 If $\varphi_t : T^1M \longrightarrow T^1M$ is a unit tangent flow then locally $T\varphi_t$ can be represented as :

$$T\varphi_t : TT^1U \longrightarrow T(\varphi_t T^1U) : (x,v,dx \oplus \delta v) \longmapsto (x(t),v(t),X(t) \oplus \nabla_v X)$$

where X(t) is a vectorfield along the sub-flow line x(t) in M such that $X(0) = dx$, $\nabla_{v(0)} X = \delta v$.

Proof: Suppose $\varphi_t : T^1U \longrightarrow T^1\varphi_t U : (x,v) \longmapsto (f_t(x,v), g_t(x,v))$ then by definition of a flow $f_0 = g_0 = id.$ and $d/dt(f_t)(x,v) = g_t(x,v)$ by the previous lemma.

$T\varphi_t : (x,v,e_1 \oplus e_2 + \Gamma(x)(v,e_1))$

$(f_t(x,v), g_t(x,v), Df_t(x,v)(e_1,e_2) \oplus Dg_t(x,v)(e_1,e_2) + \Gamma(x)(g_t, Df_t))$

If we now fix x,v,e_1,e_2 and let t vary, then:

(i) $x(t) - f_t(x,v)$ is a sub flow line in M and hence $v(t) = g_t(x,v) = \dot{x}(t)$.

(ii) $dx(t) = Df_t(x,v)(e_1,e_2)$ and $dv(t) = Dg_t(x,v)(e_1,e_2)$, then by definition of covariant co-ordinates for TT^1M, App A, putting $\delta v(t) = dv(t) + \Gamma(x(t))(v(t),dx(t))$ then $dx(t) \oplus \delta v(t) \in T_{x(t)}M \oplus T_{x(t)}M$, so that dx, δv are vectorfields along x(t)

(iii) For fixed t we can now write

$$T\varphi_t : (x(0),v(0),dx(0) \oplus \delta v(0)) \longmapsto (x(t),v(t),dx(t) \oplus \delta v(t))$$

Now
$$\begin{aligned}\nabla_{v(t)}dx &\equiv d/dt(dx) + \Gamma(x(t))(v(t),dx(t)) \\ &= d/dt(Df_t(x,v)(e_1,e_2)) + \Gamma(x)(v,dx) \\ &= D(d/dt(f_t))(x,v).(e_1,e_2) + \Gamma(x)(v,dx) \\ &= Dg_t(x,v)(e_1,e_2) + \Gamma(x)(v,dx) \\ &= dv(t) + \Gamma(x)(v,dx) = \delta v(t).\end{aligned}$$

q.e.d.

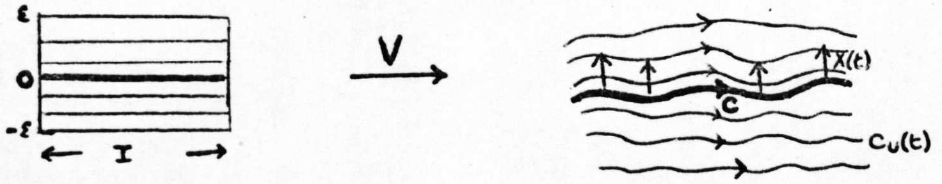
We would expect the behaviour of $T\varphi_t$ to be reflected in the flow lines and sub-flow lines.

Definition: (Milnor[22]) Given a curve $c : I \to M$ then a variation of c is a C^∞ map $V : I \times I \longrightarrow M$ such that

(i) $c_u(t) = V / I \times \{u\}$ is a smooth map for all u.

(ii) $c_0(t) = V / I \times \{0\} = c(t)$ " " " t.

and $X(t) = \left.\dfrac{\partial V}{\partial u}\right|_{\substack{u=0\\t=t}}$ is the variation vectorfield along c.



Definition: Given a sub-flow line c of a unit tangent flow then a Variation through flow lines is a variation as above such that $c_u(t)$ is a sub-flow line for all u.

Proposition 3:3. A unit tangent flow $\varphi_t : T^1M \to T^1M$ has $T\varphi_t$ locally of the form:

$$T\varphi_t : (x,v,X \oplus \nabla_v X) \longrightarrow (x(t),v(t),X(t) \oplus \nabla_{v(t)} X)$$

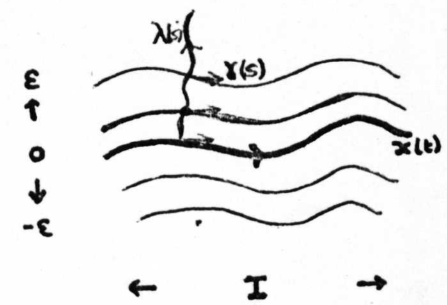
where x(t) is a sub-flow line and X(t) the variation vectorfield of a variation through sub-flow lines.

Proof: Given $(x,v,e_1 \oplus e_2 + \Gamma(x)(v,e_1)) \in T_{(x,v)}T^1M$ consider any curve $\gamma : I \to T^1M$ st. $\gamma(0) = (x,v)$ and $\dot{\gamma}(0) = (e_1,e_2)$ ie. $T\gamma(0,1) = (x,v,e_1,e_2)$

If $Z : T^1M \to TT^1M$ generates the flow φ_t then

$\lambda = (\pi_1 \circ \gamma) : I \to M$ is a curve in M with vectorfield $\dot{P}(s) = \gamma(s)$ along $\lambda(s)$.

Consider the subflow lines in M generated by Z thus:-



$V : I \times (-\varepsilon,\varepsilon) \longrightarrow M$

$(t,u) \longmapsto (\pi_1 \circ \varphi_t \circ \gamma)(u)$

then $V(t,0) = x(t)$.

Fix $t = \tau$, and consider $V(\tau,u) = W_\tau(u)$; then

$$\left.\frac{\partial V}{\partial u}\right|_{\substack{u=0\\t=\tau}} = \left.\frac{dW}{du}\right|_{u=0} \equiv TW_\tau(0,1) = T\pi_1(x(t),v(t),X(t) \oplus \nabla_{v(\tau)}X)$$

$$= (x(\tau),\ X(\tau))$$

Hence X is a variation vectorfield.

q.e.d.

Notation: (i) If $Z : T^1M \to TT^1M : (x,v) \longmapsto (x,v,v \oplus Y(x,v))$ is a unit tangent flow let $\tilde{Z} : T^1M \to TM : (x,v) \mapsto (x, Y(x,v))$ ie. $\tilde{Z} = K \circ Z$.

(ii) For a <u>k</u>-flow let $N_1 : T^1M \to T^1M : (x,v) \mapsto (x, N_1(x,v))$ ie. $\tilde{Z} = k_1 N_1$.

<u>Lemma 3:4</u>. If Z is a unit tangent flow then the variation vector field of a variation through sub-flow lines satisfies:

$$\nabla_v^2 X + R(v,X)v = \nabla_X \tilde{Z}.$$

Proof: Given a sub-flow line of Z, consider a variation through flow lines as in the previous proposition with X as variation vectorfield. We can regard V as a parameterised surface in M and apply the results of section 0.

In particular there are two special vectorfields:

(i) $X(t,u) = \dfrac{\partial V}{\partial u}(t,u)$ such that $X(t) = X(t,0)$.

(ii) $v(t,u) = \dfrac{\partial V}{\partial t}(t,u)$ such that $v(t) = v(t,0)$.

Recall that if Y is a vectorfield along a curve $c(t)$ then $\nabla_{\dot c(t)} Y \equiv \dfrac{DY}{dt}$ (Notation)

If V is a variation through sub-flow lines then for fixed u $c_u(t) = V(t,u)$ is a sub-flow line in M with velocity $v(t,u)$.

Now $Z : T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus \nabla_v v)$ since Z is a vectorfield. Hence

$$\frac{D}{dt}\left(\frac{\partial V}{\partial t}\right) = \tilde Z(V(t,u),v(t,u)) \text{ and so } \tilde Z(t,u) \text{ is a}$$

vectorfield on V.

$$\nabla^2_{v(t)} X = \frac{D}{dt}\left(\frac{D}{dt}\frac{\partial V}{\partial u}\right)(t,0) = \frac{D}{dt}\left(\frac{D}{du}\frac{\partial V}{\partial t}\right) \text{ by Prop}^n \text{ 0:1}$$

$$= \frac{D}{du}\left(\frac{D}{dt}\frac{\partial V}{\partial t}\right)(t,0) + R\left(\frac{\partial V}{\partial u}, \frac{\partial v}{\partial t}\right)\frac{\partial v}{\partial t}\Bigg|_{\substack{t=t\\u=0}} \text{ by Prop}^n \text{ 0:1}$$

$$= \frac{DZ}{du}\Bigg|_{\substack{t=t\\u=0}} + R(X,v)v$$

Since $\dfrac{D}{du}\Big|_{\substack{u=0\\t=t}} \equiv \nabla_{X(t)}$ we have

$$\nabla^2_v X + R(v,X)v = \nabla_{X(t)} Z$$

q.e.d.

Corollary 3:3:1. For the geodesic flow $\nabla_v^2 X + R(v,X)v = 0$. ie. Jacobi Fields.

Proof: $Z : (x,v) \longrightarrow (x,v,v \oplus 0)$

Note. The variation V above is not necessarily unique, and so neither is the 'induced' vectorfield $\tilde{Z}$. However if we used another variation V' then since $\nabla_X \tilde{Z}$ ($\nabla_X \tilde{Z}'$) is the covariant derivative of $\tilde{Z}$ ($\tilde{Z}'$) along $W(u) = V(t,u)$ ($W'(u) = V'(t,u)$) we use the result that the covariant derivative of a vectorfield along a curve is independent of any extension to see that we get the same result with either variation.

$\tilde{Z}$ is a vectorfield on a surface V, and hence depends on X, the variation vectorfield. This is useful for later manipulations, but to generate an equation for X, it is more convenient to give it an invariant meaning.

Recall the connector map $K : TT^1M \longrightarrow T^1M : (x,v,e\oplus f) \longmapsto (x,f)$

Lemma 3:5. If X is the variation vectorfield of a variation through sub-flow lines then:

$$\nabla_{X(t)} \tilde{Z} = (K \circ TZ)(X \oplus \nabla_v X).$$

Proof: Consider the variation surface V and the induced vectorfield $\tilde{Z}(u,v) = \tilde{Z}(V(u,t), v(u,t))$

$$\nabla_{X(t)} Z \equiv \left.\frac{D\tilde{Z}}{du}\right|_{u=0} = \left.\frac{d\tilde{Z}}{du}\right|_{u=0} + \Gamma(x(t))(X(t), \tilde{Z}(V(t,0), v(t,0)))$$

$$= D_x Z . \left.\frac{\partial V}{\partial u}\right|_{u=0} + D_v Z . \left.\frac{\partial v}{\partial u}\right|_{u=0} + \Gamma(x)(X,Z) \text{ by chain rule.}$$

where $D_x Z \equiv \left(\frac{\partial Z^i}{\partial x^j}\right)$, $D_v Z \equiv \left(\frac{\partial Z^i}{\partial v^j}\right)$ are the Jacobian matrices

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in $T\tilde{Z}$.

Hence $\nabla_{X(t)}\tilde{Z} = D_x\tilde{Z}.X(t) + D_v\tilde{Z}(\frac{d}{dt}\frac{\partial V}{\partial u}) + \Gamma(x)(X,\tilde{Z})$ since $v \equiv \frac{\partial V}{\partial t}$

$$= D_x\tilde{Z}.X(t) + D_v\tilde{Z}(\nabla_v X - \Gamma(x)(v,X)) + \Gamma(x)(X,\tilde{Z})$$

since $\frac{d}{dt}\frac{\partial V}{\partial u} = \frac{dX}{dt} = \nabla_v X - \Gamma(x)(v,X)$

$$= (KoT\tilde{Z})(x,v, X \oplus \nabla_v X).$$

This follows since:

$(x,v,e \oplus f) \longmapsto (x,v,e,f-\Gamma(x)(v,e)) \longmapsto$

$(x,\tilde{Z}(x,v),e,D_xN.e + D_vN\,(f-\Gamma(v,e))) \longmapsto$

$(x,\tilde{Z}(x,v),e \oplus D_xN.e + D_vN(f-\Gamma(v,e)) + \Gamma(x)(\tilde{Z}(x,v), e)).$

q.e.d.

The previous few lemmas show:

<u>Proposition 3:6</u>. If $\varphi_t : T^1M \to T^1M$ is a unit tangent flow generated by a vectorfield Z then

(i) Any variation through sub-flow lines satisfies

$$\nabla_v^2 X + R(v,X)v = \nabla_X \tilde{Z}$$ where $\tilde{Z}$ is an induced vectorfield on the variation surface.

(ii) $T\varphi_t : TT^1M \to TT^1M$ is locally of the form

$$(x,v,X(0) \oplus \nabla_{v(0)} X) \longmapsto (x(t),v(t),X(t) \oplus \nabla_{v(t)} X)$$

where X is a variation vectorfield satisfying the global equations:-

$$\nabla_v^2 X + R(v,X)v = (KoT\tilde{Z})(X \oplus \nabla_v X).$$

Since we are primarily interested in <u>k</u>-flows where $\tilde{Z} = k_1 N_1$ we define the following:

<u>Definition</u>: The variation vectorfield of a variation through $\underline{k}$-lines is a $\underline{k}$-field satisfying:-

$$\nabla_v^2 X + R(v,X)v = k_1 \nabla_X N_1 = k_1(K o T N_1)(X \oplus \nabla_v X).$$

<u>Corollary 3:6:1</u>. If φ_t is a $\underline{k}$-flow then locally $T\varphi_t$ is of the form:-

$$:(x,v,X \oplus \nabla_v X) \longmapsto (x(t),v(t),X(t) \oplus \nabla_{v(t)} X) \text{ for}$$

some $\underline{k}$-field X.

<u>Corollary 3:6:2</u>. For a $\underline{k}$-flow $(T\varphi_t o \tilde{Z})$ can be represented as $(x,v,v \oplus k_1 N_1) \longrightarrow (x(t),v(t),v(t) \oplus k_1 N_1(t))$, the zero variation.

Proof: $v \oplus k_1 N_1$ satisfies $\nabla_v^2 X + R(v,X)v = k_1 \nabla_X N_1$ since $R(v,v)v \equiv 0$. If we construct the variation as previously yielding $v \oplus k_1 N_1$ then $\gamma(t) \equiv (x(t),v(t))$, $\lambda(t) = x(t)$, so all that is generated is the $\underline{k}$-line itself.

q.e.d.

Section 4: Matrix methods in the study of k-flows.

In this section we concentrate on k-flows and show that the statements concerning vectorfields along k-lines can be converted into a convenient form, using the parallel frames developed earlier to gain matrix differential equations. Presumably these techniques can be adapted for arbitary unit tangent flows if at each point of a sub-flow line there is a basis for $T_m M$ which alters in a known way.

Given a k-line $c : I \longrightarrow M$ we have seen that at each point we have a basis for $T_{c(t)}M$ by completing an n-frame at $c(0)$, $(v(0), N_1(0), \dots, N_r(0), E_1(0), \dots, E_s(0))$ and parallel translating the E_i along c to form a basis at each point, with the geodesic normals.

Notation: $F(t) = (v(t), \dots, E_s(t)) = (F_0(t), \dots, F_{n-1}(t))$ say, so a vectorfield along the k-line can be written as

$$X \equiv X^i(t) F_i(t) \equiv \begin{bmatrix} X_0(t) \\ X_1(t) \\ \vdots \\ X_{n-1}(t) \end{bmatrix}$$

Consider
$$\begin{aligned}
\nabla_{v(t)} X = \nabla_v (X^i F_i) &= (X^i)' F_i + X^i \nabla_v F_i . \qquad (\ ' = d/dt) \\
&= (X^i)' F_i + (X^0 k_1 F_1 + X^1 (k_2 F_2 - k_1 F_1) + \dots \\
&\quad + X^r (-k_r F_{r-1}) + X^{r+1} \nabla_v E_1 + \dots + X^s \nabla_v E_s) \\
&= (X^i)' + (X^0 k_1 F_1 + \dots + X^r (-k_r F_{r-1}) + 0)
\end{aligned}$$
since $\nabla_v E_i \equiv 0$ by construction.

So in matrix notation we have

$$\nabla_v X \equiv \frac{d}{dt} \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

or

$$\nabla_v X \equiv \frac{dX}{dt} + \nabla . X \text{ where } \nabla = \begin{array}{c} \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right] \end{array} \Big\updownarrow n$$

and
$$A = \underbrace{\begin{bmatrix} 0 & -k_1 & 0 & & 0 & 0 & 0 \\ k_1 & 0 & -k_2 & & 0 & 0 & 0 \\ 0 & k_2 & 0 & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & -k_{r-1} & 0 \\ 0 & 0 & 0 & \cdots & k_{r-1} & 0 & -k_r \\ 0 & 0 & 0 & \cdots & 0 & k_r & 0 \end{bmatrix}}_{r+1}$$

Notation: $(d/dt + \nabla)^s X \equiv \frac{d^sX}{dt^s} + \ldots + \binom{s}{r} \nabla^{s-r} \frac{d^rX}{dt^r} + \ldots + \nabla^s X$

where we have matrix multiplication.

Lemma 4:1 Given a vectorfield X along a $\underline{k}$-line c then $\nabla^m_{v(t)} X = (d/dt + \nabla)^m X$.

Proof: By induction.

Holds for m=1 by above, so assume for m=r.

Let Y be the matrix for $\nabla^r_v X$ then by assumption

$$Y^i = X_i^{(r)} + \ldots + \binom{r}{s} (\nabla^{r-s})_{ij} X_j^{(s)} + \ldots + (\nabla^r)_{ij} X_j$$
$$= \sum_{s=0}^{r} \binom{r}{s} (\nabla^{r-s})_{ij} X_j^{(s)} \quad \text{where } X^{(s)} = \frac{d^sX}{dt^s}$$

Hence $\nabla^{r+1}_v X = \nabla_v(\nabla^r_v X) = (d/dt + \nabla)(Y)$

$$(\nabla_v^{r+1})^i = \sum_{s=0}^{r} \left\{ \binom{r}{s} (\nabla^{r-s})_{ij} X_j^{(s+1)} + \binom{r}{s} \nabla_{ip} (\nabla^{r-s})_{pq} X_q^{(s)} \right\}$$

$$= \sum_{s=0}^{r} \left\{ \binom{r}{s} (\nabla^{r-s})_{ij} X_j^{(s+1)} + \binom{r}{s} (\nabla^{r-s+1})_{ij} X_j^{(s)} \right\}$$

$$= \sum_{s=0}^{r} \left\{ \binom{r}{s-1} (\nabla^{r-s-1})_{ij} X_j^{(s)} + \binom{r}{s} (\nabla^{r-s+1})_{ij} X_j^{(s)} + X_i^{(r+1)} \right\}$$

$$= \sum_{s=1}^{r} \left\{ \left(\binom{r}{s-1} + \binom{r}{s}\right) (\nabla^{r-s+1})_{ij} X_j^{(s)} + X_i^{(r+1)} \right\} + (\nabla_{ij}^{r+1}) X_j$$

$$= \sum_{s=0}^{r+1} \left\{ \binom{r+1}{s} (\nabla^{r+1-s})_{ij} X_j^{(s)} \right\}$$

$$= \left\{ \left({}^{d}\!/\!{}_{dt} + \nabla \right)^{r+1} X \right\}_i$$

q.e.d.

Note: Along a geodesic $\nabla \equiv 0$ so $\nabla_v^m X \equiv \dfrac{d^m X}{dt^m}$.

Later we shall be particularly interested in:

<u>Corollary 4:1:1</u>.

$$\nabla_v^2 X \equiv \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}'' + 2\left[\begin{array}{c|c} 0 & -k_1\ 0 \cdots 0 \\ \hline k_1 & \\ 0 & \\ \vdots & A \\ 0 & \end{array}\right] \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}' + \left[\begin{array}{c|c} -k_1^2 & 0\ k_1k_2\ 0 \cdots 0 \\ \hline 0 & \\ k_1k_2 & \\ 0 & B \\ \vdots & \\ 0 & \end{array}\right] \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

$$\text{where } A = \left[\begin{array}{cccccc|c} 0 & -k_2 & 0 & \cdots & 0 & 0 & \\ k_2 & 0 & -k_3 & \cdots & 0 & 0 & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \underset{\sim}{0} \\ 0 & 0 & 0 & \cdots & 0 & -k_r & \\ 0 & 0 & 0 & \cdots & k_r & 0 & \\ \hline & & & & & & 0 \\ & & & \underset{\sim}{0} & & & \ddots \\ & & & & & & 0 \end{array}\right] \quad B = \left[\begin{array}{cccccc|c} -k_2^2-k_1^2 & 0 & k_2k_3 & & & & \\ 0 & -k_2^2-k_3^2 & 0 & & & & \\ k_2k_3 & 0 & -k_3^2-k_4^2 & & & & \underset{\sim}{0} \\ & & & \ddots & & & \\ & & & & -k_{r-1}^2-k_r^2 & 0 & \\ & & & & 0 & -k_r^2 & \\ \hline & & & \underset{\sim}{0} & & & \underset{\sim}{0} \end{array}\right]$$

Matrix notation is also useful in simplifying expressions containing the Riemann metric.

Given column matrices X,Y we have:-

(i) a matrix norm $|X(t)| = (\Sigma X_i^2)^{\frac{1}{2}}$

(ii) an inner product $X.Y = X_i Y_i$

Lemma 4:2 Given two vectorfields, X,Y, along a k-line then $\|X(t)\| = |X(t)|$ and $\langle X(t),Y(t)\rangle = X.Y$ where $\| \;\|$ and $\langle , \rangle$ are the Riemann norm and inner product.

Proof: $\|X(t)\|^2 = \langle X(t),X(t)\rangle = \langle X^i F_i, X^j F_j\rangle$

$= X^i X^j \langle F_i, F_j\rangle = \Sigma(X^i)^2$

since $\langle F_i , F_j\rangle = 0$ for $i \neq j$

$$\langle X(t),Y(t)\rangle = \langle X^i F_i, Y^j F_j\rangle = X^i Y^j \delta^i_j = X^i Y^i.$$

q.e.d.

In particular this also simplifies the expression for the norm of $T\varphi_t$ in TT^1M, under the norm in the fibres $T_v T^1 M$, induced from G:- Appendix A.

Corollary 4:2:1 If X is a k-field along a k-line then

$$\|T\varphi_t(x,v,X \oplus \nabla_v X)\|^2 = \|x(t),v(t),X(t) \oplus \nabla_{v(t)} X\|^2_{v(t)}$$

$$= \|X(t)\|^2_{x(t)} + \|\nabla_{v(t)} X\|^2_{x(t)} = |X(t)|^2 + |\nabla_v X(t)|^2.$$

Lemma 4:3 For a vectorfield along a k-line then

$R(v,X)v \equiv \begin{bmatrix} & R & \end{bmatrix}\begin{bmatrix} X \end{bmatrix}$ where R is of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R' & \\ 0 & & & \end{bmatrix}$$

and R' is a symmetric matrix.

Proof:

In matrix notation

$$\langle R(v,X)v\rangle \equiv \begin{bmatrix} \langle R(F_0,X^iF_i)F_0, F_0\rangle \\ \langle R(F_0,X^iF_i)F_0, F_1\rangle \\ \vdots \\ \langle R(F_0,X^iF_i)F_0, F_{n-1}\rangle \end{bmatrix} \qquad \text{since } v = F_0$$

$$= \begin{bmatrix} \Sigma X^i\langle R(F_0,F_i)F_0, F_0\rangle \\ \Sigma X^i\langle R(F_0,F_i)F_0, F_1\rangle \\ \vdots \\ \Sigma X^i\langle R(F_0,F_i)F_0, F_{n-1}\rangle \end{bmatrix} \qquad \text{by linearity of } R$$

$$= \begin{bmatrix} \langle R(v,v)v,F_0\rangle & \langle R(v,v)v,F_1\rangle \cdots \langle R(v,v)v,F_{n-1}\rangle \\ \langle R(v,v)v,F_1\rangle & \\ \vdots & \langle R(F_0,F_i)F_0,F_j\rangle \\ \langle R(v,v)v,F_{n-1}\rangle & \end{bmatrix}\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{bmatrix}$$

Now $R(v,v)v \equiv 0$ and $\langle R(Z,W)X,Y\rangle = \langle R(X,Y)Z,W\rangle$ Prop^n 0:2

So $\langle R(F_0,F_i)F_0,F_0\rangle = 0$ and

$R'_{ij} = \langle R(F_0,F_i)F_0,F_j\rangle = \langle R(F_0,F_j)F_0,F_i\rangle = R'_{ji}$

Hence R' is symmetric.

q.e.d.

Corollary 4:3:1 If X is a vectorfield along a k-line then $X^T\underline{R}X = K(X,v)\,|\mathrm{nor}X|^2$, where $K(X,v)$ is the sectional curvature of the space spanned by X,v and NorX is the matrix representing the part of X normal to the curve.

Proof: $[X_0,\ldots,X_{n-1}]\begin{bmatrix} & \underline{R} & \end{bmatrix}\begin{bmatrix} X_0 \\ \vdots \\ X_{n-1}\end{bmatrix} = \langle R(F_0,X^iF_i)F_0,X^jF_j\rangle$

$= \langle R(v,X)v,X\rangle = K(X,v)\|X\wedge v\|^2$ by def^n of sectional curvature.

where $\|X\wedge v\|$ is the area of the parallelogram spanned by X,v.

$$\|X\wedge v\|^2 = \|X\|^2\|v\|^2 - \langle X,v\rangle^2$$
$$= \{|X|^2 - X_0^2\} = \sum_{i=0}^{n-1} X_i^2 - X_0^2 = \sum_{i=1}^{n-1} X_i^2 = |\mathrm{nor}X|^2$$

q.e.d.

Corollary 4:3:2 If X is aperpendicular vectorfield along a $\underline{k}$-line then $X^T \underline{R} X = K(X,v)|X|^2$

Proof X = norX.

Corollary 4:3:3 If M has strict negative/positive curvature then R is negative/positive definite. (semi-definite if R is not strict)

Proof. If $X^T = (X_1,\dots,X_{n-1})$ then $X^TRX = Y^T\underline{R}Y$ where $Y^T = (0,X_1,\dots,X_{n-1})$

So $X^TRX = K(Y,v)|Y|$ by last Corollary

$= K(Y,v)|X|$

When $X \neq 0$ and M has curvature <0 then $K(Y,v)<0$, and hence $X^TRX < 0$ since $|X|>0$

When $X = 0$ $X^TRX = 0$

implies R is negative definite. etc

q.e.d.

The above properties are the basis for proving the Anosov properties for the geodesic flow on manifolds of strict negative curvature, for then it can be shown (Anosov[2],[3]) that perpendicular Jacobi fields satisfy the matrix differential equation:

$$Y'' + RY = 0$$

and the negative definiteness of R guarantees the necessary hyperbolic properties of the Jacobi fields, which span a subspace of TT^1M. It is our intention to construct similar equations for k-flows in general.

We have to convert the expression $\nabla_X N_1$ into matrix notation, using the normals, and obtaining something of the form $AX' + BX$ say. To do this it is expedient to use the idea of the variation surface and use the theorems of section O.

If we are given a k-field X along a k-line, c, generated by a variation V say then we can split X into components normal to the k-line and tangential to it.

Definition: If $X(t) = \sum_{j=0}^{n-1} X^j F_j$ is a k-field along a k-line then: (i) $Y(t) = \sum_{j=1}^{n-1} X^j F_j$ is the (associated) normal k-field and (ii) $X_0(t) = X^0 F_0(t)$ the (associated) tangential k-field.

Hence $X(t) = Y(t) + X_0(t)$, both components being tangential to the variation surface V.

Lemma 4 : 4. $\nabla_{X_0(t)} N_1 \equiv A_1 X(t)$ in matrix notation.

Proof: $\nabla_{X_0(t)} N_1 = \nabla_{X^0 F_0} N_1 = X^0 \nabla_v N_1 = X^0(k_2 N_2 - k_1 v)$

$= X_0(t)(k_2 F_2(t) - k_1 F_0(t))$ by Frenet,

and the fact that $N_1 \equiv F_1$ along the k-line c.

So $$\nabla_{X_0(t)} N_1 \equiv \begin{bmatrix} -k_1 & & \\ 0 & & \\ k_2 & \underline{0} & \\ \vdots & & \\ 0 & & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix} = A_1\underline{X} \text{ say.}$$

q.e.d.

<u>Lemma 4:5</u>. $$\nabla_Y N_1 \equiv \begin{bmatrix} 0 & (A_2)_{ij} \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}' + \begin{bmatrix} 0 & (A_3)_{ij} \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

where $(A_2)_{ij} = \langle K\circ TN_1(x,v,0 \oplus F_j),\ F_i\rangle$

$(A_3)_{ij} = \langle K\circ TN_1(x,v,F_j \oplus {}_vF_j),\ F_i\rangle$

$i=0,\dots,n-1;\ j=1,\dots,n-1.$

Proof; Setting $Y = \sum_{j=1}^{n-1} X^j F_j$ tangential to the variation surface we have:

$$\begin{aligned}
\nabla_Y N_1 &= (K\circ TN)(x,v,\ Y \oplus \nabla_v Y) \quad \text{as in Lemma 3:5} \\
&= D_xN(x,v)(Y) + D_vN(x,v)(\nabla_v Y - \Gamma(v,Y)) + \Gamma(x)(N,Y) \\
&= D_xN.\sum_{j=1}^{n-1} X^j F_j + D_vN(\dot{Y}^j F_j + (A_{jk}Y^k)F_j) \\
&\quad - D_vN(\Gamma(v,Y)) + \Gamma(N,Y) \quad \text{by Lemma 4:1} \\
&= \sum_{j=1}^{n-1} X^j D_xN.F_j + D_vN.(A_{jk}Y^k)F_j - \sum_{j=1}^{n-1} X^j D_vN.\Gamma(v,F_j) + \\
&\quad \sum_{j=1}^{n-1} X^j\,\Gamma(N,F_j) + \sum_{j=1}^{n-1} \dot{Y}^j D_vN.F_j. \quad \text{by linearity of D and } \Gamma. \\
&= \sum_{j=1}^{n-1} \dot{X}^j D_vN.F_j + \sum_{j=1}^{n-1} X^j (D_xN.F_j - D_vN.\Gamma(v,F_j) + \Gamma(N,F_j)) + \\
&\quad D_vN(x,v)(X_1(k_2F_2 - k_1F_0) + \dots + (k_{i+1}F_{i+1} - k_iF_{i-1})X_i + \dots \\
&\quad \dots + (-k_rF_{r-1})).
\end{aligned}$$

$$
\text{So}\ \nabla_Y N \equiv \left[\begin{array}{c|c} 0 & \begin{array}{l}\langle D_v N.F_j, F_i\rangle \\ j=1,\ldots,n-1 \\ i=0,\ldots,n-1\end{array}\end{array}\right]\left[\begin{array}{c} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{array}\right]' +
$$

$$
\left[\begin{array}{c|c} 0 & \begin{array}{l}\langle D_x N.F_j - D_v N.\Gamma(v,F_j) + \Gamma(N,F_j) + D_v N.\nabla_v F_j\ ,\ F_i\rangle \\ j=1,\ldots,n-1;\ i=0,\ldots,n-1\end{array}\end{array}\right]\left[\begin{array}{c} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{array}\right]
$$

$$
\equiv \left[\begin{array}{c|c} 0 & \begin{array}{l}\langle KoTN(x,v,0 \oplus F_j), F_i\rangle \\ j=1,\ldots,n-1 \\ i=0,\ldots,n-1\end{array}\end{array}\right]\left[\begin{array}{c} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{array}\right]' +
$$

$$
\left[\begin{array}{c|c} 0 & \begin{array}{l}\langle KoTN(x,v,F_j \oplus \nabla_v F_j), F_i\rangle \\ j=1,\ldots,n-1 \\ i=0,\ldots,n-1\end{array}\end{array}\right]\left[\begin{array}{c} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{array}\right]
$$

q.e.d.

<u>Proposition 4:6</u>.

$$
\nabla_X N \equiv \left[\begin{array}{c|cccc} 0 & -1 & 0 & \ldots & 0 \\ \hline 0 & & & & \\ 0 & & D & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right]\left[\begin{array}{c} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{array}\right]' + \left[\begin{array}{c|cccc} -k_1 & 0 & k_2 & \ldots & 0 \\ \hline 0 & & & & \\ k_2 & & E & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right]\left[\begin{array}{c} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{array}\right]
$$

where $D_{ij} = \langle KoTN(x,v,0 \oplus F_j), F_i\rangle \quad i,j=1,2,\ldots,n-1$

$E_{ij} = \langle KoTN(x,v,F_j \oplus \nabla_v F_j), F_i\rangle \quad i,j=1,2,\ldots,n-1$

Proof: By lemmas 4:4 and 4:5 wehave:-

$$\nabla_X N = \nabla_Y N + \nabla_{X_0} N$$

$$\equiv \left[\begin{array}{c|c} 0 & A_2 \end{array}\right]\begin{bmatrix} X_0 \\ \vdots \\ X_{n-1}\end{bmatrix}' + \left[\begin{array}{c|c} \begin{matrix}-k_1 \\ 0 \\ k_2 \\ \vdots \\ 0\end{matrix} & A_3 \end{array}\right]\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{bmatrix}$$

Consider the first row of this matrix expression; this gives the value of $\langle \nabla_X N_1, F_0\rangle = \langle \nabla_X N_1, v\rangle$. Now consider N_1 as a vectorfield on the variation surface V, and apply Lemma 0:3, about vectorfields on surfaces;

$k_1\langle \nabla_X N_1, v\rangle = \langle \nabla_X \nabla_v v, v\rangle$ using Frenet formulae along c

$= \langle \nabla_v \nabla_X v, v\rangle + \langle R(v,X)v, v\rangle$ lemma 0:3

$= \langle \nabla_v \nabla_X v, v\rangle + \langle R(v,v)v, X\rangle$ lemma 0:2

$= \langle \nabla_v \nabla_X v, v\rangle + 0$ since $R(X,X)X \equiv 0$

On V $\langle v,v\rangle \equiv 1$ so that $0 = X\langle v,v\rangle = 2\langle \nabla_X v, v\rangle$ lemma 0:1

$\langle v, \nabla_X v\rangle \equiv 0$ implies that $0 = v\langle v, \nabla_X v\rangle$ lemma 0:1

$= k_1\langle N_1, \nabla_X v\rangle + \langle v, \nabla_v \nabla_X v\rangle$

So $k_1\langle \nabla_X N, v\rangle = \langle \nabla_v \nabla_X v, v\rangle = -k_1\langle N_1, \nabla_v X\rangle$

By lemma 4:1 $\nabla_v X \equiv \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1}\end{bmatrix}' + \begin{bmatrix} 0 & -k_1 & 0 & \cdots & 0 \\ k_1 & 0 & -k_2 & \cdots & 0 \\ 0 & k_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1}\end{bmatrix}$

and $\langle \nabla_v X, N_1\rangle$ is the second row of this:-

$$\equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} X_0 \\ \vdots \\ X_{n-1}\end{bmatrix}' + \begin{bmatrix} k_1 & 0 & k_2 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} X_0 \\ \vdots \\ X_{n-1}\end{bmatrix}$$

So $\nabla_X N =$

$$\begin{bmatrix} 0 & -1 \quad 0 \quad . \quad . \quad 0 \\ \vdots \\ 0 & (A_2)_{ij} \\ \vdots & i,j=1,..,n-1 \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}' + \begin{bmatrix} -k_1 & 0 \quad k_2 \quad 0 \ldots 0 \\ 0 & \\ k_2 & (A_3)_{ij} \\ \vdots & i,j=1,..,n-1 \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

q.e.d.

<u>Proposition 4:7.</u> If X is a <u>k</u>-field along a <u>k</u>-line then the normal <u>k</u>-field Y satisfies:-

$$Y'' + (2B-k_1D)Y' + (R+T-k_1E)Y = 0$$

and the tangential <u>k</u>-field:-

$$X_0' = k_1X_1 .$$

Proof: By Corollary 4:1:1, lemma 4:3 , Prop^n 4:6

$$\nabla_v^2 X + R(v,X)v - k_1\nabla_X N$$

$$\equiv \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}'' + 2\begin{bmatrix} 0 & -k_1 \quad . \quad . \quad 0 \\ k_1 & \\ \vdots & B \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}' + \begin{bmatrix} -k_1^2 & 0 \quad k_1k_2 \quad . \quad . \quad 0 \\ 0 & \\ k_1k_2 & C \\ \vdots & \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & -k_1 \quad 0 \ldots . \quad 0 \\ 0 & \\ \vdots & k_1D \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}' - \begin{bmatrix} -k_1^2 & 0 \quad k_1k_2 \quad . \quad . \quad . \quad 0 \\ 0 & \\ k_1k_2 & k_1E \\ \vdots & \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} X \end{bmatrix}'' + \begin{bmatrix} 0 & -k_1 \quad 0 \quad . \quad . \quad . \quad 0 \\ 2k_1 & \\ 0 & 2B - k_1D \\ \vdots & \\ 0 & \end{bmatrix} \begin{bmatrix} X \end{bmatrix}' + \begin{bmatrix} 0 & 0 \quad . \quad . \quad 0 \\ 0 & \\ \vdots & C + R - k_1E \\ 0 & \end{bmatrix} \begin{bmatrix} X \end{bmatrix}$$

We know that $X \oplus \nabla_v X \in T_v T^1 M$ so $\nabla_v X \perp v$ see Appendix A

and $v \equiv \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$

so $X_0' - k_1 X_1 = 0$ lemma4:1 ——————— A

Consider the second line of the matrix expression for the k-field:-

$$X_1'' + 2k_1 X_0' + (2B - k_1 D)_{2j} X_j' + (C + R - k_1 E)_{2j} X_j = 0$$

Substituting A above:-

$$X_1'' + (2B - k_1 D)_{2j} X_j' + (C + R - k_1 E)_{2j} + 2k_1^2 X_1 = 0$$

Hence the matrix expression becomes:-

$$\begin{bmatrix} X_0 \\ X_1 \\ \\ \\ X_{n-1} \end{bmatrix}'' + \begin{bmatrix} 0 & -k_1 \quad 0 \ldots 0 \\ 0 & \\ \cdot & 2B - k_1 D \\ \cdot & \\ 0 & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \cdot \\ \cdot \\ X_{n-1} \end{bmatrix}' + \left\{ \begin{bmatrix} 0 & 0 \quad \cdot \quad \cdot \quad \cdot \quad 0 \\ 0 & \\ \cdot & C + R - k_1 E \\ \cdot & \\ 0 & \end{bmatrix} + \begin{bmatrix} 0 & 0 \ldots 0 \\ 0 & 2k_1^2 \\ \cdot & \quad 0 \\ \cdot & \\ 0 & \end{bmatrix} \right\} \begin{bmatrix} X_0 \\ X_1 \\ \cdot \\ \cdot \\ X_{n-1} \end{bmatrix}$$

or splitting the expression and setting:-

$$Y \equiv \begin{bmatrix} X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}$$

then the above becomes:-

$$X_0' - k_1 X_1 = 0$$

$$Y'' + (2B - k_1 D) Y' + (R + T + k_1 E) Y = 0$$

where:-

$$D_{ij} = \langle K\circ TN(x,v,0 \oplus F_j),F_i\rangle$$

$$E_{ij} = \langle K\circ TN(x,v,F_j \oplus \nabla_v F_j),F_i\rangle$$

$$B = \left[\begin{array}{ccccc|c} 0 & -k_2 & \cdots & 0 & 0 & \\ k_2 & 0 & \cdots & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \underset{\sim}{0} \\ 0 & 0 & \cdots & 0 & -k_r & \\ 0 & 0 & \cdots & k_r & 0 & \\ \hline & & \underset{\sim}{0} & & & \underset{\sim}{0} \end{array}\right]$$

$$T = \left[\begin{array}{cccccccc|c} k_1^2-k_2^2 & 0 & k_2k_3 & \cdots & 0 & 0 & 0 & & \\ 0 & -k_2^2-k_3^2 & 0 & \cdots & 0 & 0 & 0 & & \\ k_2k_3 & 0 & -k_3^2-k_4^2 & \cdots & 0 & 0 & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \underset{\sim}{0} \\ & & & & -k_{r-1}^2-k_{r-2}^2 & 0 & k_rk_{r-1} & & \\ 0 & 0 & 0 & \cdots & 0 & -k_{r-1}^2-k_r^2 & 0 & & \\ 0 & 0 & 0 & \cdots & k_rk_{r-1} & 0 & -k_r^2 & & \\ \hline & & & & \underset{\sim}{0} & & & & \underset{\sim}{0} \end{array}\right]$$

q.e.d.

Notes: (i) In the geodesic case $\underline{k} = \underline{0}$ we get

$Y'' + RY = 0$ for the normal part and

$X_0'(t) \equiv 0$ for the tangential part. So we have a Jacobi field, and the usual result that a Jacobi vectorfield once perpendicular remains perpendicular.

(ii) When $Y \equiv 0$ then $X_1 = 0$ and hence $X(t) = av(t)$ is a solution for a=constant, corresponding to the zero variation.

Variation theory for the geodesic case has been extensively studied, particularly in the context of Morse theory, and the study of geodesic flows on manifolds of negative curvature, (Milnor [22] and Anosov [2])

It is hoped that these preliminary ideas and the results of Appendix B showing that $\underline{k}$-lines arise naturally in mechanics in a similar way to geodesics, might result in an analogous theory for $\underline{k}$-lines, for instance are $\underline{k}$-lines the minimal energy curves of some energy function?

Up to now we have looked at a single $\underline{k}$-line and one particular frame of reference along it, formed by parallel translating an arbitary completion at $c(0)$. However suppose we start with another set of vectors $\{E_1',\dots,E_s'\}$ at $c(0)$ and form another n-frame along the curve, $F'(t) = (v(t),N_1(t),\dots,N_r(t),E_1'(t),\dots,E_s'(t))$, do we get the same results as using the n-frames:-
$F(t) = (v(t),N_1(t),\dots,N_r(t),E_1(t),\dots,E_s(t))$?

<u>Proposition 4:8</u>.(Invariance of notation)
Let $X(t)$ be a vectorfield along a $\underline{k}$-line with matrix representation $Y(t)$ wrt. $F(t)$ and $Z(t)$ wrt $F'(t)$ then $Z(t) = \underline{\theta}\, Y(t)$ where $\underline{\theta}$ is an orthogonal matrix of the form

$$\begin{bmatrix} I_{r+1} & 0 \\ 0 & \theta \end{bmatrix}$$

Proof: Consider $F(t)$ and $F'(t)$ agreeing in the first $(r+1)$ directions, the common geodesic normals. Hence for each j, $E_j'(0)$ lies in the $(E_1(0),\dots,E_s(0))$ subspace of $T_{c(0)}M$.

If we write $E_j'(0) = \sum_{p=1}^{s} \langle E_j'(0),E_p(0)\rangle\, E_p(0)$

$$= \underline{\theta}_{jp}(0)E_p(0) \text{ say}$$

$E_j'(t)$ and $E_p(t)$ are formed by parallel translation which preserves the inner product: so that $\langle E_j'(t), E_p(t)\rangle = \langle E_j'(0), E_p(0)\rangle$ for all t.

$$\text{So} \quad E_j'(t) = \sum_{p=1}^{s} \langle E_j'(t), E_p(t)\rangle E_p(t) = \sum_{p=1}^{s} \langle E_j'(0), E_j(0)\rangle E_p(t)$$

$$= \theta_{jp} E_p(t) \quad \text{for constant } \theta.$$

$$\text{Now } Z_j(t) = \langle X(t), E_j'(t)\rangle = \langle X(t), \theta_{jp} E_p(t)\rangle$$

$$= \theta_{jp} \langle X(t), E_p(t)\rangle = \theta_{jp} Y_p(t)$$

since θ is constant.

Since $F(t)$ agrees with $F'(t)$ in the first $(r+1)$ places then the above shows that:

$$\begin{bmatrix} Z_0(t) \\ Z_1(t) \\ \vdots \\ Z_{n-1}(t) \end{bmatrix} = \begin{bmatrix} I_{r+1} & 0 \\ 0 & \langle E_p'(0), E_p(0)\rangle \end{bmatrix} \begin{bmatrix} Y_0(t) \\ Y_1(t) \\ \vdots \\ Y_{n-1}(t) \end{bmatrix}$$

Doing the opposite calculation we have:

$$Y(t) = \underline{\theta}' Z(t) \quad \text{with } \underline{\theta}' = \begin{bmatrix} I_{r+1} & 0 \\ 0 & \langle G_p(0), E_j(0)\rangle \end{bmatrix}$$

Hence $\underline{\theta}^{-1} = \underline{\theta}' = \underline{\theta}^T$ and $\underline{\theta}$ is thus orthogonal.

q.e.d.

The fact that $\underline{\theta}$ is a constant matrix allows us to show that the simple form for $\nabla_v X$ is preserved.

Proposition 4:9. The form of the matrix representation $\nabla_v X \equiv [X]' + \nabla.[X]$ is independent of the frames.

Proof: If as in the last Prop^n we have Y as the matrix representation of X wrt. the frame F(t) and Z wrt. F'(t), then by above:

$Z = \underline{\theta} Y$ and $Z' = \underline{\theta} Y'$ since θ is constant.

Moreover $Z = \underline{\theta}Y = Y$ since

$$= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \underline{\theta} = \begin{bmatrix} I_{r+1} & 0 \\ 0 & \theta \end{bmatrix}.$$

Hence $\nabla_v X \equiv Y' + \nabla.Y = Z' + \nabla.Z.$

q.e.d.

In particular we should like to know the form of the equations for the normal $\underline{k}$-fields as we change the frames.

Definition: Two second order equations $Y'' + BY' + CY = 0$ and $Y'' + SY' + TY = 0$ are orthogonally equivalent if $B = \underset{\sim}{Q}S\underset{\sim}{Q}^T$ and $C = \underset{\sim}{Q}T\underset{\sim}{Q}^T$ for an orthogonal matrix $\underset{\sim}{Q}$. ie. B,C are orthogonally equiv. to S,T resp.

Proposition 4:10. Given two Frames F, F' along a $\underline{k}$-line then the normal $\underline{k}$-fields satisfy orthogonally equivalent equations wrt. these frames.

Proof: Using the same representatives as before for a vectorfield X along a $\underline{k}$-line, then if X is a normal $\underline{k}$-field then if in frames F(t) Y satisfies:-

$Y'' + (2B-k_1D)Y' + (R+T-k_1E)Y = 0$ then let X using frames $F'(t)$ satisfy:-

$Z'' + (2B_1-k_1D_1)Y' + (R_1+T_1-k_1E_1)Y = 0$

By the form of B,T as of the form $\begin{bmatrix} \square & 0 \\ 0 & 0 \end{bmatrix}$

and $\underline{\theta}$ of the form $\begin{bmatrix} I_{r+1} & 0 \\ 0 & \theta \end{bmatrix}$

then $B_1 = B = \theta B_1 \theta^T$ $T_1 = T = \theta T_1 \theta^T$.

Now $(R_1)_{ij} = \langle R(v,E_j')v, E_i' \rangle = \langle R(v, \theta_{jp}E_p)v, \theta_{ip}E_p \rangle$

$= \theta_{jp} \langle R(v,E_p)v, E_q \rangle \theta_{iq}$

$= \theta_{jp} R_{pq} \theta_{iq}$ by linearity of R

implies that $R_1 = \underline{\theta} R \underline{\theta}^T$

$(D_1)_{ij} = \langle KoTN(0 \oplus F_j'), F_i' \rangle = \langle KoTN(0 \oplus \theta_{jp}F_p), \theta_{iq}F_q \rangle$

$= \theta_{jp} \langle KoTN(0 \oplus F_p), F_q \rangle \theta_{iq}$

$= \theta_{jp} D_{qp} \theta_{iq}$

implies $D_1 = \underline{\theta} D \underline{\theta}^T$

$(E_1)_{ij} = \langle KoTN(F_j' \oplus \nabla_v F_j'), F_i' \rangle = \langle KoTN(\theta_{jp}F_p \oplus \theta_{jp}F_p), \theta_{iq}F_q \rangle$

by the proposition above.

$= \theta_{iq} E_{pq} \theta_{jp}$ as above, by linearity of KoTN.

implies that $E_1 = \underline{\theta} E \underline{\theta}^T$.

So the matrix equations satisfied by X can be written:-

$Y'' + (2B-k_1D)Y' + (R+T-k_1E)Y = 0$ or

$Z'' + \theta(2B-k_1D)\theta^T Y' + \theta(R+T-k_1E)\theta^T Y = 0$

for an orthogonal matrix θ.

q.e.d.

CHAPTER TWO. ON THE HYPERBOLIC PROPERTIES OF SELF-ADJOINT k-FLOWS.

Introduction.

Anosov (2& 3) in studying the geodesic flow on manifolds of negative curvature showed they satisfied a certain 'condition U'. Since then 'condition U' flows have been studied in abstract, yielding a rich source of results in both measure theory and dynamical systems.

Definition: A flow $\varphi_t : N \to N$ is an Anosov flow (satisfies condition U) if:

(i) $Z(n) \neq 0 \quad \forall n \in N$ where Z is the v.f. generating the flow.

(ii) For each $n \in N$ there is a splitting into a Whitney sum:

$$T_n N = E_n^s \oplus E_n^u \oplus Z_n$$

into subspaces st. $\dim E_n^s, \dim E_n^u \neq 0$ and Z_n is the one dimensional subspace generated by $Z(n)$ and:

$$\text{(a)} \quad \forall \xi \in E_n^s \quad \|T\varphi_t(\xi)\| \leqslant a\|\xi\|\, e^{-ct} \quad \forall t \geqslant 0$$

$$\forall \eta \in E_n^u \quad \|T\varphi_t(\eta)\| \geqslant b\|\eta\|\, e^{ct} \quad \forall t \geqslant 0$$

$$\text{(b)} \quad \forall \xi \in E_n^s \quad \|T\varphi_t(\xi)\| \geqslant b\|\xi\|\, e^{-ct} \quad \forall t \leqslant 0$$

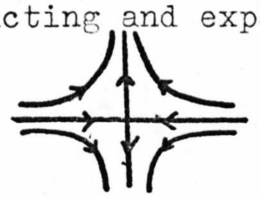
$$\forall \eta \in E_n^u \quad \|T\varphi_t(\eta)\| \leqslant a\|\eta\|\, e^{ct} \quad \forall t \leqslant 0$$

where $a, b, c > 0$ are constants independent of t, ξ, η.

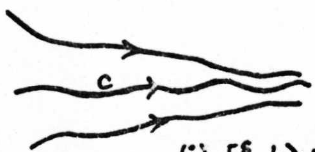
Condition (i) says that the flow has no equilibrium points; as is the case for a k-flow on T^1M, since all points are in motion. This is not true for the extended k-flow on TM, since Z/zero section $\equiv 0$.

We have seen in the last section that the variation through k-lines reflects the behaviour of

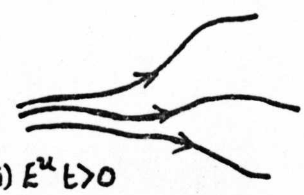
$T\varphi_t$. The space Z_n corresponds to the zero variation, while we can interpret the splitting transversal to Z_n into a contracting and expanding subspace;



as corresponding to a family of k-lines (variation) positively asymptotic to a given k-line, and another family negatively asymptotic to the line.



(i) E^s $t>0$



(ii) E^u $t>0$

In particular Anosov ([2]) showed that the geodesic flow is Anosov on manifolds with curvature $K < 0$, whereas the flow om $\mathbb{R}^n$ is not; and in general the flow on manifolds of positive curvature is also not Anosov. Hence it would be convenient if given a manifold M we could assign to any k-flow on M a number depending on the k_i and the normals N_i; ie. the configuration of the external field; say K_0 such that if the manifold has curvature less than this 'Anosov number' the flow is Anosov. In this way we can judge whether the hyperbolic nature of a manifold of negative curvature is 'strong' enough to counter the spread of the flowlines caused by the k-field.

Section: 5. Self-Adjoint k-flows.

Due to the limitations of matrix theory used later we now have to restrict our attention to a certain class of k-flows. Also we consider M to be a compact manifold, or any manifold on which the k-lines are complete. ie. defined on $(-\infty, \infty)$.

To consider the hyperbolic properties of k-flows, note that $N = T^1M$ in the definition of Anosovity, and hence for each $v \in T^1M$ we have to construct a splitting:

$$T_vT^1M = E_v^s \oplus E_v^u \oplus Z_v.$$

This we shall hope to do by constructing for each k-line a family of k-fields each of which tends to 0 as $t \to \infty$ corresponding to the family of k-lines positively assymptotic to the k-line. Similarly for the reverse direction. Since we require these k-fields to be transversal to Z_v for each v, this corresponds to the k-fields having non-zero normal parts, hence we look at the normal k-fields. First we look at the matrix equations derived in the last section, and as usual begin by converting a second order equation into two first order ones.

Definition: A k-flow is self-adjoint if its normal k-field satisfies matrix equations of the form:

$$\begin{aligned} Y' &= A(t)Y + Z \\ Z' &= -C(t)Y - A^T(t)Z \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix}' = \begin{bmatrix} C & A^T \\ A & I \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix}$$

where C and hence $\begin{bmatrix} C & A^T \\ A & I \end{bmatrix}$ is symmetric.

that is the normal k-fields satisfy a Hamiltonian matrix equation.

Recall that the normal $\underline{k}$-fields satisfy:-

$$Y'' + (2B - k_1 D(t))Y' + (R(t) + T - k_1 E(t)) = 0 .$$

This means that we have to impose restrictions on the coefficient matrices. Work has been done on non-selfadjoint matrix equations, particularly by Kreith (29), who obtained a Prüfer transformation for solutions. However at present it is not developed enough to give the results we shall need.

Proposition 5:1. A $\underline{k}$-flow is self-adjoint iff along any $\underline{k}$-line the matrix D is skew-symmetric and $D' - E + E^T = 0$.

Proof: Suppose the normal $\underline{k}$-fields satisfy:

$$\left.\begin{aligned} Y' &= A(t)Y + Z \\ Z' &= -C(t)Y - A^T Z \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (1).$$

then $Y'' + (A^T - A)Y' + (C(t) - A'(t) - A^T A)Y = 0$

SO the flow is self-adjoint iff

(i) $2B - k_1 D(t) = A^T - A$ (ii) $C = (R + T + A^T A + A' - k_1 E)$

$A^T - A$ is skew-symmetric and hence $D(t)$ must be skew, since B is of the form

$$\begin{bmatrix} 0 & -k_2 & 0 & \\ k_2 & 0 & -k_3 & \\ 0 & k_3 & 0 & \\ & & & \text{etc} \end{bmatrix}$$

and is already skew symmetric.

We can represent any matrix D as $\frac{1}{2}(D + D^T) + \frac{1}{2}(D - D^T) = P + Q$ say where P is symmetric and Q skew-symmetric.

Hence we require $\frac{1}{2}(D + D^T)$, and so

$$\begin{aligned} A^T - A &= 2B - \tfrac{1}{2}k_1(D - D^T) \\ &= (B - B^T) + \tfrac{1}{2}(D^T - D) \quad \text{since B is skew.} \\ &= (B + \tfrac{1}{2}k_1 D^T) - (B^T + \tfrac{1}{2}k_1 D) \end{aligned}$$

Let $A = B^T + \frac{1}{2}k_1 D$ and so $A' = \frac{1}{2}k_1 D'$ since B is constant.

This makes $C = R + T + A' - k_1 E + (B^T + \frac{1}{2}k_1 D)^T (B^T + \frac{1}{2}k_1 D)$

$$= R + T + \tfrac{1}{2}k_1 D' - k_1 E + BB^T + \tfrac{1}{2}k_1 (BD + (BD)^T) + \tfrac{1}{4}k_1^2 D^T D$$

$$= R + (T + B^T B) + k_1 (\tfrac{1}{2}D' + \tfrac{1}{2}(BD + (BD)^T) + \tfrac{1}{4}k_1 D^T D - E)$$

$$= R + M + k_1 N \text{ say.}$$

since (a) R contains information about the manifold's curvatu[re]

(b) M " terms only in the k_i: the flow's curvatu[re]

(c) N " information about the normals, N_i.

Now $N = \frac{1}{2}D'(t) - E + \frac{1}{2}(BD + D^T B^T) + \frac{1}{4}k_1 D^T D$

$$= \tfrac{1}{2}D'(t) - \tfrac{1}{2}(E + E^T) - \tfrac{1}{2}(E - E^T) + \tfrac{1}{2}(BD + D^T B^T) + \tfrac{1}{4}k_1 D^T D.$$

$$= \tfrac{1}{2}(D' - E + E^T) - \tfrac{1}{2}(E + E^T) + \tfrac{1}{2}(BD + D^T B^T) + \tfrac{1}{4}k_1 D^T D$$

$$= \tfrac{1}{2}(D' - E + E^T) + \text{symmetric terms.}$$

Now R is symmetric (Prop^{n}4:3), M is symmetric (see def^n of T and $B^T B$ is symmetric). D' is skew and so is $E - E^T$, so we require $D'(t) - E + E^T = 0$, since the zero matrix is the only one which is both symmetric and skew.

q.e..d.

<u>Notation</u>: The above shows that a self-adjoint <u>k</u>-flow has normal <u>k</u>-fields satisfying:

$$Y' = (B^T + \tfrac{1}{2}k_1 D)Y + Z$$

$$Z' = -(R + M + k_1 N)Y - (B^T + \tfrac{1}{2}k_1 D)^T Z.$$

where $N = \frac{1}{2}k_1 D' + \frac{1}{2}(BD + D^T B^T) + \frac{1}{4}k_1^2 D^T D$ and

$M =$

$$T + B^T B$$

$$= \begin{bmatrix} k_1^2 & 0 & \cdots & 0 \\ 0 & & & \\ 0 & & 0 & \\ \cdot & & & \\ 0 & & & \end{bmatrix}$$

We can remove the skew symmetry of D from the conditions if we consider:

Definition: A $\underline{k}$-flow is linear if the map $N_1 : T^1M \longrightarrow T^1M$ is given by a linear transformation.

Examples: (i) $\underline{k}$-flows on oriented surfaces since $N_1 \mid T^1_mM$ is represented by $(v_1, v_2) \longmapsto \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(ii) Any $\underline{k}$-flow on an even dimensional manifold having $N_1 : (m)(v_1, \ldots, v_{2n}) \longmapsto \begin{bmatrix} 0 & \pm I_n \\ \mp I_n & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{2n} \end{bmatrix}$.

They are interesting because:

Proposition 5:2. A linear $\underline{k}$-flow is not reversible and preserves the Riemann volume element.

Proof: Recall that a flow is reversible if given a flow line, c, then parameterising in the reverse direction we still have a flow line (geodesic flow). If we have a linear $\underline{k}$-flow then given a $\underline{k}$-line, c, then $\nabla_v v = k_1 N_1$ and so reversing the direction $\nabla_{-v}(-v) = \nabla_v v = k_1 N_1$ But $-c$ should be a flow line with normals given by N_1, however N_1 is linear so $N_1(-v) = -N_1(v)$ so $\nabla_{-v}(-v) = -k_1 N_1$ not $k_1 N_1$. (If we reverse a $\underline{k}$-line on the Lobachevsky plane with right hand normals we obtain one with left hand normals)

If N_1 / T^1_mM is linear then it can be represented by $N_1 / T^1_mM : (x,v) \longmapsto (x, (N_1(x))v)$ for a matrix $N_1(x)_{ij}$

Now D_vN is the derivative of N_1 within the fibre T^1_mM and so $D_vN(x,v) = N_1(x)$ since the flow is linear.

$$\text{trace}.D_vN_1(x,v) = \sum \text{eigenvalues of } D_vN_1$$

If λ is a non-zero eigenvalue of D_vN_1 then $\exists$ e st $D_vN_1.e = \lambda e$

By the above $D_vN(x).e = N_1(x).e = N_1(e) \perp e$ by definition

Hence D_vN has no non-zereo eigenvalues, and $\mathrm{tr}D_vN \equiv 0$.

q.e.d.

Proposition 5:3. If φ_t is a linear k-flow then the variation matrix $D = [\langle KoTN(x,v,0 \oplus F_j), F_i\rangle]$ is skew symmetric.

Proof: $D_{ij} = \langle KoTN(0 \oplus F_j), F_i\rangle = \langle D_vN.F_j, F_i\rangle$.

We can represent any matrix C as $\frac{1}{2}(C + C^T) + \frac{1}{2}(C - C^T) = P + Q$ say

$$2P_{ij} = \langle D_vN.F_j , F_i\rangle + \langle D_vN.F_i, F_j\rangle$$

$$= \langle D_vN.(F_j+F_i),(F_j+F_i)\rangle - \langle D_vN,F_j,F_j\rangle - \langle D_vN.F_i,F_i\rangle$$

by linearity of $\langle , \rangle$, and D_vN.

$$= 0$$

since if φ_t is linear $D_vN \equiv N$ and $\langle N_1(e),e\rangle = 0$.

q.e.d.

Proposition 5:4. The k-flows on an oriented surface of constant negative curvature are self-adjoint and the normal k-fields satisfy $y'' - (K^2 - 2k_1^2)y = 0$.

Proof. The normal k-fields satisfy scalar equations, which are trivially self-adjoint,

$N(x,v)$ can be represented as $:(x_1,x_2,v_1,v_2) \longmapsto (x_1,x_2,\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix})$

Hence $D_xN \equiv 0$ and $D_vN(e_1,e_2) = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} = N(e_1,e_2)$

$$D = \langle D_vN . N , N\rangle = \langle \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}, N\rangle = \langle v,N\rangle = 0$$

So $D' = 0$

$$E = \langle D_X N.N_1 + D_V N(\nabla_V N) - D_V N(\Gamma(v,N_1) + \Gamma(N_1,N_1)),\ N_1)\rangle \quad \text{lemma4:6}$$

$D_X N \equiv 0$ and $D_V N.(\nabla_V N_1) = k_1 D_V N(v) = -k_1 N_1(v)$.

By the conditions for a <u>k</u>-flow (Example on p27)

$$D_V N.(\Gamma(v,N_1)) = D_V N.(D_V N(\Gamma(v,v))) = N^2(\Gamma(v,v)) = -\Gamma(v,v).$$

$$\Gamma(N_1,N_1) = \left\{ \tfrac{1}{2A}\,[-v_2, v_1] \begin{bmatrix} x & y \\ y & -x \end{bmatrix} \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix},\ \tfrac{1}{2A}\,[-v_2, v_1] \begin{bmatrix} -y & x \\ x & y \end{bmatrix} \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} \right\}$$

$$= \tfrac{1}{2A}((v_2^2 - v_1^2)x - 2v_1v_2y,\ -2v_1v_2x + (v_1^2 - v_2^2)y)$$

$$= -\Gamma(v,v).$$

So $E = \langle 0 - k_1N_1 + 0,\ N_1\rangle = -k_1$

$R = \langle R(v,N_1)v, N_1\rangle = -K^2$, the curvature of the manifold.

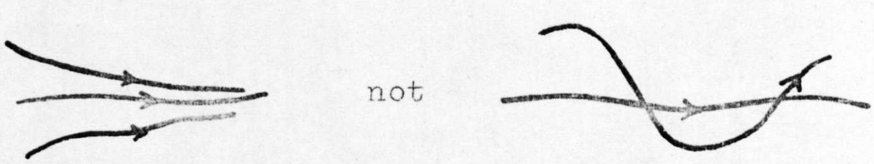
Hence y satisfies:

$$y' = z,\quad z' = (K^2 - 2k_1^2)y.$$

q.e.d.

We next review some ideas from the theory of matrix differential equations.

Klingenberg ([15]) has shown that any manifold whose geodesic flow is Anosov has no conjugate points, as do manifolds of strict negative curvature. This implies that along any geodesic there are no non-trivial Jacobi fields that disappear more than once. In particular the result applies to perpendicular Jacobi fields, which correspond to our normal <u>k</u>-fields. Intuitively this means we are studying variations of the form:



While we have no direct analogue for the present of Klingenberg's theorem we have motivation for:

<u>Definition</u>: A Riemann manifold is disconjugate for a $\underline{k}$-flow if there are no non-trivial normal $\underline{k}$-fields along any $\underline{k}$-line of the flow, Y say, that disappear more than once. ie. $\nexists t_1, t_2$ st. $Y(t_1) = Y(t_2) = 0$.

Notes. (1) Along any $\underline{k}$-line this reduces to the normal definition of disconjugacy for the matrix equation satisfied by the normal $\underline{k}$-field, that there are no non trivial solutions (Y,Z) st. $Y(t_1)=Y(t_2)=0$ for some $t_1 \neq t_2$.

(2) We shall later show that disconjugacy, as in the geodesic case, is automatic for manifolds of negative enough curvature.

(3) On $\mathbb{R}^n$ or S^{n-1} the $\underline{k} = (k,0,\ldots,0)$ flows are not disconjugate, for the $\underline{k}$-lines are closed circles (Section 1) and given any circle pick out two points on the circle, construct a family of circles of the same radius through the points, the variation thus obtained having two zero points, where all the circles cross.

Along each $\underline{k}$-line of a given flow we are trying to construct an n-1 family of $\underline{k}$-fields to span E^S, ie. n-1 linearly independent solutions of the normal $\underline{k}$-field eqn. We can simplify the problem of checking linear independence by instead of considering column matrix equations for the normal $\underline{k}$-fields we consider $(n-1)\times(n-1)$ matrix equations of the form:-

$$\left.\begin{aligned} U' &= AU + V \\ V' &= -CU - A^T V \end{aligned}\right\} \quad \ldots\ldots (2)$$

Linear independence then reduces to considering solutions of (2), (U,V) say, which satisfy $\det(U(t)) \neq 0$ for all t, these giving an $n-1$ family of normal k-fields $Y(t) = \{U(t)c : c \text{ is aconstant column matrix}\}$

Definition(Hartman): (1) For any two matrix solutions (U_1,V_1), (U_2,V_2) say the constant matrix $W(U_1,U_2) = U_1^T(t)V_2(t) - V_1^T(t)U_2(t)$, for all t.

(2) Two solutions are conjugate if $W(U_1,U_2) = 0$.

(3) A solution (U,V) is self conjugate (isotropic, cojoined) if $W(U,U) = 0$

Note. If we do not have $C(t)$ symmetric then we lose the constant nature of the Wronskian, which is used later in many places.Hence we can only consider self adjoint flows at present.

To make our construction we shall use a particular solution.

Proposition 5:5. If M is disconjugate for a k-flow, and $(U_o(t),V_o(t))$ is the self conjugate solution of (2) st $U_o(0) = 0$, $V_o(0) = I$, then $\det U_o(t) \neq 0$ for all $t \gtrless 0$.

Proof: If $\det U_o(t) = 0$ for $t \neq 0$, then the columns of $U_o(t)$ are linearly dependent and there is a column matrix c st. $U_o(t)c = 0$. Hence $Y(s) = U_o(s)c$ is a solution of the column matrix equation st $Y(0) = Y(t) = 0$, contradicting M disconjugate.

q.e.d.

We shall use this later to construct a solution for which $\det U(t)$ is never zero. We also need the equation in another form:-

<u>Lemma 5:6</u>. (Copell[6] p49)
If (U,V) is a solution of the above which is invertible for all t, then $P(t) = V(t)U^{-1}(t)$ is a solution of the matrix Ricatti equation:

$$P' + PA + A^TP + P^2 + C = 0.$$

$P(t)$ is symmetric iff (U,V) is self conjugate.

To enable us to discuss disconjugacy we need some ideas from Copell ([6]). Given the equation above letting $a<b$, $z(t)$ is a piece-wise continuous vector valued function on $[a,b]$, and $y(t)$ is a solution of $y' = Ay + z$ then M is disconjugate for the <u>k</u>-flow iff

$$Q = \int_a^b (z^Tz - y^TCy)dt > 0 \quad \text{for all } a,b \in \mathbb{R}, \forall z,y.$$

This result follows immediately from the result in Copell since $I > 0$. This gives the following sufficiency condition:-

<u>Lemma 5:7</u>. M is disconjugate for a self adjoint <u>k</u>-flow if $C=(R + M + k_1N) < 0$.

Proof. $-C > 0$ implies $-\int_b^a C(t)dt > 0$ where integration is component-wise.

Suppose we work on manifolds of strict negative curvature so that $R < 0$, Cor4:4:3, then we can ask if R has enough negative definiteness to make the matrix $C < 0$ and hence disconjugacy

Notation: Given two symmetric (nxn) matrices A,B then

$A \leqslant B$ if $B-A \geqslant 0$. ie. $B - A$ is positive semi-definite.

Lemma 5:8. $M \leqslant k_1^2 I$.

Proof: $$M = \begin{bmatrix} k_1^2 & 0 & \cdot & \cdot & 0 \\ 0 & & & & \\ \cdot & & & & \\ \cdot & & 0 & & \\ \cdot & & & & \\ 0 & & & & \end{bmatrix}$$

To determine a similar bound on the matrix N we need the results on matrix norms in section 0.

Lemma 5:9. If N is a symmetric matrix there is an $n^2 \geqslant 0$ st. $N \leqslant n^2 I$.

Proof: Let $N = (N_{ij})$ $i,j=1,..,r$ $\underline{x} = (x_1,\dots,x_r)$ then

$$\underline{x}^T N \underline{x} = \sum_{i,j=1}^{r} x_i N_{ij} x_j = \sum_{i=1}^{r} N_{ii} x_i^2 + 2 \sum_{i<j=1}^{r} N_{ij} x_i x_j$$

$$= \sum_{i=1}^{r} N_{ii} x_i^2 - \sum_{i<j=1}^{r} \left(\sqrt{|N_{ij}|}\, x_i \pm \sqrt{|N_{ji}|}\, x_j\right)^2 + \sum_{i\ j=1}^{r} \left(|N_{ij}|\, x_i^2 + |N_{ij}|\, x_j^2 \right)$$

$$\leqslant - \sum_{i<j=1}^{r} \left(\sqrt{|N_{ij}|}\, x_i \pm \sqrt{|N_{ij}|}\, x_j\right)^2 + \sum_{i\leqslant j=1}^{r} N_{ij}(x_i^2 + x_j^2)$$

$$= - \sum (\quad)^2 + \sum_{i,j=1}^{r} |N_{ij}|\, x_i^2.$$

SO $\underline{x}^T(n^2 I - N)\underline{x} \geqslant \sum_{i<j=1}^{r} \left(\sqrt{|N_{ij}|}\, x_i \pm \sqrt{|N_{ij}|}\, x_j\right)^2 + \sum_{i=1}^{r} \left(n^2 - \sum_{j=1}^{r} N_{ij}\right) x_i^2$

$\geqslant 0$ when $n^2 \geqslant \underset{i=1,..,r}{\text{maximum}} \left(\sum_{j=1}^{r} |N_{ij}| \right) = \underset{j=1,..,r}{\text{maximum}} \left(\sum_{i=1}^{r} |N_{ij}| \right)$

$= |N|_5$ since N is symmetric.

q.e.d.

Corollary 5:9:1: Given any matrix N then there is an $n^2 \geqslant 0$ st $-n^2 I \leqslant N \leqslant n^2 I$.

Proof: Write $N = \frac{1}{2}(N + N^T) + \frac{1}{2}(N - N^T)$ = symmetric + skew

$\underline{x}^T N \underline{x} = \frac{1}{2} \underline{x}^T (N + N^T) \underline{x} + 0$

By above $\frac{1}{2}(N + N^T) \leqslant \frac{1}{2}(N + N^T)_5 I$, and $-\frac{1}{2}(N + N^T) \geqslant |\frac{1}{2}(N + N^T)|_5 I$

$= |\frac{1}{2}(N + N^T)|_5 \ I$.

So $\quad -|\frac{1}{2}(N + N^T)|_5 I \leqslant N \leqslant |\frac{1}{2}(N + N^T)|_5 I$.

q.e.d.

Definition: Let $p : F^n M \to F^{r+1} M : (x, v, \mathcal{J}_1, .., \mathcal{J}_{n-1}) \to (x, v, \mathcal{J}_1, .., \mathcal{J}_r)$ for $r<n-1$, and $N_{\underline{k}}$ be a $\underline{k}$- section of a $\underline{k}$- flow, then:-

(i) an n-frame $F(v) \in F^n M$ completes $N_{\underline{k}}(v)$ at v if $p \circ F(v) = N_{\underline{k}}(v)$.

ie. they agree in the first r+1 places.

(ii) A section $F : T^1 M \to F^n M$ completes the $\underline{k}$-section if $p \circ F = N_{\underline{k}}$.

Corollary 5:9:2. For all $v \in T^1 M$ and any $\underline{k}$-flow giving variation matrix N as before then there is a number $n^2(v)$ st for any completion of $N_{\underline{k}}(v)$ at v $\; N(v) \leqslant n^2(v) I$

Proof: Given $v \in T^1 M$ and any completion F(v) of $N_{\underline{k}}(v)$ we can form in the usual way the frame basis along the $\underline{k}$-line through v, and the matrix N. at v. By the above

$N(v) \leqslant |\frac{1}{2}(N(v) + N(v)^T)|_5 I . \leqslant \frac{1}{2}(n-1)^{\frac{3}{2}} |(N + N^T)|_2$

using Lemma 0:5

Hence we could take $n^2(v) = \frac{1}{2}(n-1)^{\frac{1}{2}} |N(v) + N(v)^T|_2$

Now suppose we take another completion $F'(v)$ at v giving a variation matrix $N'(v)$ say. Then by the work preceding Prop^n 4:10 then $N(v(t)) = \theta N'(v(t)) \theta^T$ for all t and a constant orthogonal matrix θ; and hence by Proposition 0:6 $|N(v)|_2 = |N'(v)|_2$. So we get the same value for $n^2(v)$ whichever frame completion is used.

q.e.d.

Definition: Given a $\underline{k}$-flow on a manifold M with variation matrices M,N then the flow has disconjugacy number $K_1^2 \geq 0$ if it is the least number st along any $\underline{k}$-line $M + k_1 N \leq K_1^2 I$, else the disconjugacy number is ∞.

Corollary5:9:2 above shows that the disconjugacy number exists at each point $v \in T^1M$, since any completion gives a common upper bound for N.

Examples: (i) Geodesic Flow $M=N=0$ so $K_1^2 = 0$.

(ii) For a $\underline{k} = (k)$ flow on an oriented surface $M + k_1 N = [2k^2]$ so $K_1^2 = 2k^2$.

Proposition 5:10. For a self adjoint $\underline{k}$-flow on a compact manifold M then $K_1^2 < \infty$.

Proof: Given a completion $F(v)$ at v and the matrices M,N formed along the $\underline{k}$-line. These were defined in terms of the matrices $D_{ij} = \langle KoTN(x,v,0 \oplus F_j), F_i \rangle$ and $E_{ij} = \langle KoTN(x,v,F_j \oplus \nabla_v F_j), F_i \rangle : i,j=1,..,n-1.$

The elements of D can be regarded as part of the following sequence of maps for any v and any completion at v.

$$T^1M \oplus T^1M \oplus T^1M \longrightarrow TTM \times TM \longrightarrow TTM \times TM \longrightarrow TM \oplus TM \longrightarrow \mathbb{R}$$

$$(x, v \oplus e \oplus f) \longmapsto ((x, v, 0 \oplus e), (x, f)) \longmapsto (TN(0 \oplus e), (x, f)) \longmapsto$$

$(KoTN(0 \oplus e), (x, f)) \longmapsto \langle KoTN(0 \oplus e), f \rangle_x$. This is a smooth real-valued function on a compact manifold and hence bdd.

We can treat E in a similar manner. However we have terms in $\langle F_j \oplus \nabla_v F_j \rangle$, but since the k_i are fixed then these vectors $\nabla_v N_i = k_{i+1} N_{i+1} - k_i N_{i-1}$ etc lie in the compact manifold $T^cM = \{w \in TM : \|w\| < c\}$ for some $c > 0$. We then have:-

$$T^1M \oplus T^1M \oplus T^1M \oplus T^1M \longrightarrow TTM \times TM \longrightarrow TTM \times TM \longrightarrow TM \oplus TM \longrightarrow \mathbb{R}.$$

$$(x, v \oplus e \oplus f \oplus g) \longmapsto ((x, v, e \oplus f), (x, g)) \longmapsto (TN(x, v, e \oplus f), (x, g))$$

$$\longmapsto (KoTN(x, v, e \oplus f), (x, g)) \longmapsto \langle KoTN(x, v, e \oplus f), g \rangle_x$$

where again this is a smooth map on a compact manifolds, real-valued and hence bdd.

q.e.d.

Later we shall need some estimatesfor the size of the matrices D and E:-

Corollary 5:10:1. Given a self adjoint $\underline{k}$-flow on a compact manifold then there are numbers $D, E \geq 0$ st along any $\underline{k}$-line and for any completion $|E(v)|_2 \leq E$ and $|D(v)|_2 \leq D$.

Proof: By argument above.

q.e.d.

<u>Proposition 5:11</u>. Given a compact Riemannian manifold, M, of negative curvature such that the sectional curvature at any point $\leqslant -K^2$, and a self adjoint $\underline{k}$-flow on M with disconjugacy number K_1^2 then M is disconjugate for the flow if $(K_1^2 - K^2) \leqslant 0$.

Proof: By lemma 5:7 sufficient to show $R + M + k_1 N \leqslant 0$. for any completion along any $\underline{k}$-line of M.

By Cor 4:4:3 R is negative definite and $\underline{x}^T R \underline{x} \leqslant -K^2 \|x\|^2$.

By the Prop^n above $\underline{x}^T (M + k_1 N)\underline{x} \leqslant \underline{x}^T (K_1^2 I)\underline{x} \leqslant K_1^2 \|x\|^2$
SO $\underline{x}^T (R + M + k_1 N)\underline{x} \leqslant (K^2 - K_1^2) \quad 0$

q.e.d.

Examples: (i) Any manifold of non-positive curvature is disconjugate for the geodesic flow, reduces to no conjugate points as in the study of Jacobi Fields.

(ii) An oriented surface of negative curvature $\leqslant -2k_1^2$ is disconjugate for either $\underline{k} = (k_1)$ flow.

Section 6: Construction of a contracting Subspace.

We have seen that the normal $\underline{k}$-fields are column matrix solutions of the matrix equations:-

$$\left.\begin{aligned} U' &= A(t)U + V \\ V' &= -C(t)U - A^T(t)V \end{aligned}\right\} \ldots\ldots(1)$$

We shall construct a particular solution of the above by adapting a method in Copell ([6] Chapt. 2) using in particular:

Lemma A. If (P,Q) is a self conjugate solution of (1) st. $\det P(t) \neq 0$ for all $t \geqslant 0$ then any solution of (1) is given by $U(t) = P(t)\left[M + \int_0^t P^{-1}_{(u)} P^{-T}_{(u)}\,du \,.\, N\right]$ where $W(P,U) = N$, $W(U,U) = M^TN - N^TM$. where. P^{-T} is notation for $(P^{-1})^T$

From now on we assume that we have a $\underline{k}$-flow on a manifold disconjugate for the flow.

Lemma 6:1. Along any $\underline{k}$-line α there is a normal $\underline{k}$-field Y_s st $Y_s(0) = c$ and $Y_s(s) = 0$ for any column matrix c.

Proof: Let (P,Q) be the solution of (1) st $P(-1) = 0$ and $P'(-1) = I$. then $\det P(-1) = 0$ and hence by the disconjugacy hypothesis $\det P(t) \neq 0$ for $t > 0$. (Prop^{n}5:5).

Let $U_s(t)$ be the solution of (1) coinciding on $[0,s]$ with

$$U_s(t) = P(t)\int_t^s P_u^{-1}.P_u^{-T}.du \qquad \text{ie } U_s(s) = 0;$$

Then by Lemma A above

$$U_s(t) = P(t)\int_0^s P^{-1}P^{-T}du - \int_0^t P^{-1}P^{-T}du$$

where $M = \int_0^s P^{-1}P^{-T}du.$ and $N = -I$.

Hence $U_s(t)$ is a solution of (1) and $W(U_s, U_s) = 0$, since M is symmetric and $W(P, U_s) = -I$.

At $s = -1$ $\quad -I = W(P, U_s) = -P'(-1)U_s(-1) = -U_s(-1)$

So $U_s(t)$ is the self conjugate solution of (1) st $U_s(-1) = 0$ and $U_s'(-1) = I$.

Disconjugacy says that $U_s(0)$ is invertible since $\det U_s(s) = 0$ for $s \gg 0$. Hence given any column matrix $c \neq 0$ then $Y_s(t) = U_s(t).U_s^{-1}(0).c$ is a normal k-field st $Y_s(0) = c$ and $Y_s(s) = 0$.

q.e.d.

Geometrically this can be interpreted by taking a variation X_s with normal part Y_s, say $X_s(0) = Y_s(0)$. Then for all $s \gg 0$ we have a family of k-lines starting within ε of $X(0)$ and cutting the original k-line close to $x(s)$. (All passing through $x(s)$ for geodesics).

To construct a contracting subspace we consider the limiting case as $s \to \infty$.

Lemma 6:2. Along any k-line there is a normal k-field $Y(t)$ st $Y(0) = c$ and $Y(t) = \lim_{s \to \infty} Y_s(t)$ for any c.

Proof: For $s \geqslant 0$

Setting (i) $W_0(t) = P(t)P^{-1}(0)$ then W_0 is invertible for $t \geqslant 0$.

(ii) $W_s(t) = U_s(t)U_s^{-1}(0)$ for $s \geqslant 0$. so $W_s(0) = I$, $W_s(s) = 0$

W_0 and W_s are solutions of (1) and so by lemma A we can set $W_s(t) = W_0(t)\left[M_s + S_0(t)N_s\right]$ where $S_0(t) = \int_0^t W_0^{-1}(u)W_0^{-T}(u)du$.

$W_s(0) = I$ implies that $I = I(M_s + 0) \Rightarrow M_s = I$ for all s.

$W_s(s) = 0$ implies that $N_s = -S_0^{-1}(s)$ since $W_0(s) \neq 0$, $W_0(-1)=0$

Following a similar method to Copell we show $\lim_{s\to\infty} S_0^{-1}(s)$ exists:-

For any invertible matrix A, $A^TA \geqslant 0$ so if $s_2 \geqslant s_1$

$$\int_0^{s_2} A^{-1}A^{-T}du \geqslant \int_0^{s_1} A^{-1}A^{-T}du. \qquad \text{Hartman([12] p388)}$$

So $N_s = -S_0^{-1}(s)$ is a monotonically increasing symmetric matrix function st $N_s \leqslant 0$ for all s. ($A<B<0 \Rightarrow B^{-1} \geqslant A^{-1}$ Copell [6] p41). Hence applying Riesz's Lemma (Hartman [12] p387) this implies that $\lim_{s\to\infty} N_s$ exists. Put $\lim_{s\to\infty} N_s = N$ and let

$$W(t) = W_0(t)\left[I + \int_0^t W_0^{-1}(u)W_0^{-T}(u).du\,N\right]$$

$W(0) = W_0(0) = I$ so $Y(t) = W(t)c$ is the desired normal <u>k</u>-field.

q.e.d.

Definition: A solution (U,V) is Principal if

(i) it is self conjugate and invertible on $[0,\infty)$

(ii) $\lim_{t\to\infty}\left(\int_0^t Y^{-1}(u)Y^{-T}(u)du\right)^{-1} = 0.$

<u>Lemma B</u>.(Copell [6] p41) If $(U_1,V_1),(U_2,V_2)$ are self conjugate solutions of (1), invertible for all $t \geqslant 0$ and

$$U_2 = U_1\left[M + \int_1^t \bar{U}_1^{-1}\bar{U}_1^{T}\, du.N\right] \text{ then}$$

$$\left(\int_1^t \bar{U}_2^{-1}.\bar{U}_2^{T}.du.\right)^{-1} = M^T\left[\int_1^t \bar{U}_1^{-1}\bar{U}_1^{T} du.M + N\right]$$

<u>Lemma C</u>: (Copell[6]p43) A principal solution is determined uniquely up to constant invertible factor.

<u>Definition</u>:If Y is a normal <u>k</u>-field along a <u>k</u>-line st $Y(t) = W(t)c$ then Y is a limit normal <u>k</u>-field.

<u>Lemma 6:3</u>. Along any <u>k</u>-line there is an (n-1) family of normal <u>k</u>-fields $\mathcal{Y}$ st if $Y\in\mathcal{Y}$ it is a limit normal <u>k</u>-field for any parameterisation of the <u>k</u>-line.ie. independent of the initial point on the <u>k</u>-line.

Proof: Choose a parameterisation of the <u>k</u>-line $C : I \longrightarrow M$ and form the matrix solution $W(t)$. Consider the family of <u>k</u>-fields formed by $\mathcal{Y}_c = \{Y : Y(t) = W(t)c \text{ for constant } c\}$

$W(t)$ is invertible for all $t \geqslant 0$ since:

(i) $t=0$ $W(0) = W_0(0) = I$

(ii) $t\geqslant 0$ $W_0(t)S_0(t)\left[\bar{S}_0^{1}(t) + N\right] = W(t)$

$W_0(t)$, $S_0(t)$ are invertible for all $t \geqslant 0$ since $W_0(-1) = S_0(-1) = 0$ and hence invertible for all $t\geqslant 0$ by disconjugacy.

$N = -\lim_{t\to\infty}\bar{S}_0^{1}(t)$ by def^n and $-\bar{S}_0^{1}(t) \leqslant -\bar{S}_0^{1}(s) \leqslant N$ for $t \leqslant s$ by lemma 6:2. Hence $N + \bar{S}_0^{1}(t) > 0$ and hence invertible.

So $\det W(t) \neq 0$ for $t\geqslant 0$, and $\mathcal{Y}_c = \{Y : Y = W(t)c\}$ is an n-1 family since the columns of W are independent.

By lemma A $W(W,W) = N - N^T = 0$ since N is symmetric as the limit of symmetric matrices.

By lemma B $\left(\int_0^t \bar{W}^1\bar{W}^T du\right)^{-1} = \left(\int_0^t \bar{W}_0^1\bar{W}_0^T du\right)^{-1} + N$

$$= \bar{S}_0^1(t) + N.$$

So as $t \to \infty$ $\lim_{t\to\infty} \bar{S}_0^1 + N = -N + N = 0.$

W(t) is hence a principal solution and hence by lemma C determined up to constant factor.

Now suppose we reparameterise the k-line and obtain $d(t) = c(t+\tau)$, and let $Z_s(t) = W_s(t+\tau)$, so

$$Z(t) = \lim_{s\to\infty} Z_s(t) = W(t+\tau) = \lim_{s\to\infty} W_s(t+\tau)$$

Z(t) is invertible and self conjugate.

$$\int_0^t \bar{Z}^1\bar{Z}^T du = \int_0^t \bar{W}^1(u+\tau)\,\bar{W}^T(u+\tau)du = \int_\tau^{t+\tau} \bar{W}^1\bar{W}^T dv$$

$$= \int_0^{t+\tau} \bar{W}^1\bar{W}^T dv - \int_0^\tau \bar{W}^1\bar{W}^T dv = S(t+\tau) + K \quad \text{say.}$$

So $\left(\int_0^t \bar{Z}^1\bar{Z}^T du\right)^{-1} = (\bar{S}^1(t+\tau))(I + K\bar{S}^1(t+\tau))$ $\quad \left[(M+N)^{-1} = \bar{M}^1(I+N\bar{M}^1)^{-1}\right]$

$\lim_{t\to\infty} \bar{S}^1(t) = 0$ since W is principal and hence

$$\lim_{t\to\infty}\left(\int_0^t \bar{Z}^1\bar{Z}^T du\right)^{-1} = 0.$$

Hence Z is a principal solution and by Lemma C differs from W only by a constant invertible factor. Hence $\mathcal{Y}_c \equiv \mathcal{Y}_d$ and the family of k-fields thus formed is independent of starting point.

q.e.d.

In the geodesic case this construction is the basis of all proofs of Anosovity on manifolds of negative curvature, be it by strict operator manipulation (Anosov [2], Anosov & Sinai [3], Eberlein [8]) or geometrically (Avez [5]).

Intuitively, (factually for geodesics), this corresponds to the construction of an assymptotic family of k-lines as $t \to \infty$ to a given k-line. The negative curvature of the manifold is used to show this is an exponential asymptoticity.

We have shown that if $[X_0(t), X_1(t), \ldots, X_{n-1}(t)]$ represents a k-field with normal part $Y(t)$ then $X_0(t)$ satisfies:

$$\frac{dX_0}{dt} = k_1 X_1(t) \qquad \text{by Prop}^n\ 4{:}7.$$

Definition: For a k-line $c: I \quad M$ let

$$\mathfrak{X}_c = \left\{ X : X \text{ is a } \underline{k}\text{-field on } c \text{ st } Y \in \mathcal{Y}_c \text{ and } X_0(t) = k_1 \int_t^\infty X_1(u)\,du \right\}$$

$\mathfrak{X}_c$ is an n-1 family of k-fields along c, as shown below.

<u>Definition</u>: For $v \in T^1M$ generating the k-line c through c let $E^s_v = \left\{ (x, v, X \oplus \nabla_v X) : X \in \mathfrak{X}_c \right\}$.

<u>Proposition 4:4</u>. E^s_v is n-1 dimensional subspace of $T_v T^1 M$ such that $T\varphi_t(E^s_v) = E^s_{\varphi_t(v)}$. ie flow invariant.

Proof: E^s_v is generated by k-fields along c given by the principal solution W along c and a tangential component.

The addition of tangential components to two linearly independent normal $\underline{k}$-fields does not affect the dependence and so E^s_v are n-1 families of vectors.

$$T\varphi_t : (x,v,X \oplus \nabla_v X) \longmapsto (x(t),v(t),X(t) \oplus \nabla_{v(t)} X)$$

By lemma 6:4 the normal part of the $\underline{k}$-field Y satisfies $Y(t) \in \mathcal{Y}_d$, where d as before is c reparameterised: $d(s)=c(t+s)$, and $X_0(t) = k_1 \int_t^\infty X_1(u)du$. So $X(t) \in \mathcal{X}_d$ and $(X(t) \oplus \nabla_{v(t)} X) \in E^s_{v(t)}$.

Consider conversely $(X(0) \oplus \nabla_{v(0)} X) \in E^s_{v(t)}$

$Y(0) = Z(0)c$ for some constant c and matrix Z as in lemma6:3. $X_0(0) = k_1 \int_0^\infty c_j Z_{1j}(u)du$

Reparameterising Z as in lemma6:3 to obtain W put

$$P(t) = k_1 \int_t^\infty W_{1j}(u)c_j du = W(t)c$$

then $T\varphi_t(P(0) \oplus \nabla_{v(0)} P) = X(0) \oplus \nabla_{v(0)} X$ by invariance properties of principal solutions.

q.e.d.

We begin now to make some estimates of the growth of the $\underline{k}$-fields along c. We have the Riemann metric on TT^1M given by:

$$|||(x,v,e \oplus f)|||^2_v = \|e\|^2_x + \|f\|^2_x . \quad \text{Appendix A.}$$

If we have a fixed frame $(v,N_1,\ldots,N_r,E_1,\ldots,E_s)$ and vectorfields X,Y along c then

$$|||x(t),v(t),X(t) \oplus Y(t)|||^2 = |X(t)|^2 + |Y(t)|^2$$

where $|\ |$ is the matrix norms.

Lemma 6:5. If $X(t) \in \mathcal{X}_c$ is a $\underline{k}$-field along a $\underline{k}$-line c st the normal $\underline{k}$-field Y satisfies $\|Y(t)\| \leqslant a \|Y(0)\| e^{-ct}$ for all $t \geqslant 0$ then $\|X(t)\| \leqslant b \|X(0)\| e^{-ct}$ for all $t \geqslant 0$ and constants a, b, c independent of the $\underline{k}$-line.

Proof: If $\|Y(t)\| \leqslant a\|Y(0)\| \, e^{-ct}$ then

$$|Y(t)| \leqslant a|Y(0)| \, e^{-ct} .$$

Hence $$|X_0(t)| = \left| k_1 \int_t^\infty X_1(u)du \right| \leqslant k_1 \int_t^\infty |X_1(u)| \, du$$

$$\leqslant k_1 \int_t^\infty |Y(u)| \, du \quad ak_1 |Y(0)| \int_t^\infty e^{-cu} \, du$$

$$= \frac{ak_1}{c} \|Y(0)\| \, e^{-ct}$$

Now $$|X(t)|^2 = |X_0|^2 + |Y(t)|^2$$

$$\leqslant \left(\frac{a^2 k_1^2}{c^2} + a^2 \right) |Y(0)|^2 \, e^{-2ct}$$

$$\leqslant \left(a^2 \left(\frac{k_4^2}{c^2} + 1 \right) \right) \|X(0)\|^2 \, e^{-2ct}$$

So $\|X(t)\| \leqslant b \, \|X(0)\| \, e^{-ct}$ and b is independent of the k-line.

q.e.d.

This says that the normal part of the k-field determines the hyperbolic nature of the whole k-field.

To consider $\nabla_v X$ we need the following result on principal solutions, adapted from Copell.

Lemma 6:6. If $U(t)$ is a self conjugate principal solution of (1) then $|U'(t) U^{-1}(t)| \leqslant L^2$ for constant $L^2 > 0$ independent of the k-line.

Proof: Recall that if (U,V) is a self conjugate, invertible solution of (1) then $P = V U^{-1}$ is a solution of:

$$P' + PA + A^TP + P^2 + C = 0. \quad \text{Lemma 5:6.}$$

By the work leading up to Prop 5:10 there is L_0^2 st

$$-L_0^2 I \leqslant C \leqslant L_0^2 I .$$

(i) Consider the eigenvalues of P, say ξ corresponding to λ, of unit length.

$$\xi^T P' \xi = -\xi^T C \xi - \xi^T P^2 \xi - \xi^T A^T P \xi - \xi^T PA \xi$$
$$= -\xi^T C \xi - \lambda \xi^T (A^T + A) \xi - \lambda^2 \xi^T \xi$$
$$= -\xi^T C \quad -\lambda^2 .$$
$$\leqslant L_0^2 - \lambda^2 .$$

using symmetry of P , ξ is unit and $A + A^T = 0$.

Hence $\xi^T P' \xi < 0$ when $L_0^2 - \lambda^2 < 0$ ie. $\lambda \leqslant -L_0$ or $\lambda \geqslant L_0$.

(ii) Consider a fixed time s and the interval $\left[s, s+\frac{1}{2L_0}\right]$

If μ is the smallest eigenvalue of P claim $\mu(t) \geqslant -L_0$

Suppose not and $\mu(s) \leqslant -L_0$ ie $\mu(s)^2 \geqslant L_0^2$

For small h:

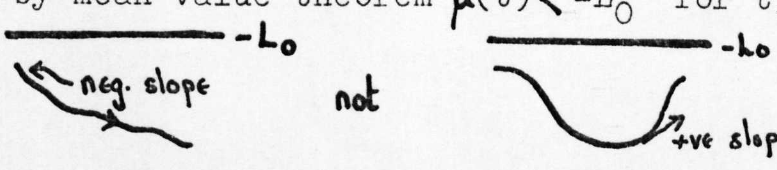
$$\xi^T P(t+h)\xi = \xi^T P(t)\xi + h\left[\xi^T P'(t)\xi + O(1)\right]$$

$$\therefore \mu(t+h) \leqslant \mu(t) + h\, O(1) \quad \text{if } \mu^2(t) \geqslant L_0^2 \text{ by (i)}$$

$$D^+\mu(t) = \lim_{h\to 0^+} \tfrac{1}{h}\left[\mu(t+h) - \mu(t)\right] < \lim_{h\ 0^+} O(1) = 0.$$

$D^+\mu(s)$ 0 by assumption, so $\mu(t)$ is a cts function st $\mu(s) \leqslant -L_0$, $D^+\mu(s) < 0$ and if $\mu(t) \leqslant -L_0$ then $D^+\mu(t) < 0$

Hence by mean value theorem $\mu(t) < -L_0$ for $t \geqslant s$.



Consider the function f(t) st $f(s) = \mu(s)$, $f'(t) = \frac{1}{2}\mu^2(t)$.

then $\frac{d^n f}{dt^n}(t) = \frac{(-1)^n}{2^n} n! \; (f(t))^{n+1}$

By Taylor's theorem:

$$f(t) = f(s) - \tfrac{1}{2}(t-s)f(s)^2 + (\tfrac{1}{2})^2(t-s)^2 f(s)^3 - (\tfrac{1}{2})^3(t-s)^3 f(s)^4 \ldots$$

$$= f(s)\left[1+\tfrac{1}{2}f(s)(t-s)\right]^{-1}$$

$$= \mu(s)\left[1+\tfrac{1}{2}\mu(s)(t-s)\right]^{-1}$$

Hence $\mu(t) \leqslant f(t) \leqslant \mu(s)\left[1+\tfrac{1}{2}\mu(s)(t-s)\right]^{-1}$

RHS $\to -\infty$ as $t \to s - 1/2\mu(s)$ $\quad \because \mu(s) < 0$ so $s - 1/2\mu(s) > s + 1/2L_0$

$\mu(t)$ is continuous on $[s, s + 1/2L_0]$ and hence bdd on the interval. So $s - 1/2\mu(s) > s + 1/2L_0$

$\mu(s) > -L_0$ contradicting hypothesis.

(iii) A similar arguement on the largest eigenvalue, $\nu(t)$, arguing by contradiction shows that $\nu(t) \leqslant L_0$.

Hence $P(s)$ has eigenvalues in the range $[-L_0, L_0]$ for all s

Since P is symmetric $|P|_3 = \max \{\text{eigenvalues}\} = \max[|\mu(t)|, |\nu(t)|]$

$$\leqslant L_0$$

Now by lemma0:5 $|P| \leqslant (n-1)^{\frac{3}{2}} L_0 + |A|$

$\leqslant L^2$ since A is bounded; Cor 5:10: 1

Moreover all of these estimates are independent of the particular <u>k</u>-line.

q.e.d.

We use this estimate in the proof of a lemma adapted from Eberlein, which for completeness we give in full.

<u>Lemma 6:7</u>. Given any <u>k</u>-line and any $R > 0$, then there is $t_0 > 0$ st $\|Y(t)\| > R\,\|Y'(0)\|$ for $t > t_0$ and any normal <u>k</u>-field Y st $Y(0) = 0$.

Proof: Let (P,Q) be the matrix solution of (1) st $P(0) = 0, P'(0) = I$. Multiplying by a constant we can assume

$\|Y'(0)\| = 1$ since $Y(t) = P(t)c$ for some constant c.
By disconjugacy $P(t)$ is self conjugate and invertible for $t \geq 0$.

$$\text{Let} \quad M(t) = \int_t^{\infty} P^{-1} P^{-T} du \quad \text{for } t \geq 0$$

By differentiation and substitution $D(t) = P(t)M(t)$ is a self conjugate solution of (1). $P^{-1}P^{-T} > 0$ so $M(t)$ and hence $D(t)$, is invertible for $t > 0$.

$$\text{Let } Q = P' - A(t)P \qquad E = D' - A(t)D$$

$$V = ED^{-1} \qquad U = QP^{-1}$$

$$(U-V)(t) = P^{-T}(t)M^{-1}(t)P^{-1}(t)$$

P,D are self conjugate solutions, since $P(0) = D(0)$
By lemma5:6 U,V are symmetric solutions of the matrix Ricatti equation. So for a unit vector $x \in \mathbb{R}^{n-1}$:

$$\begin{aligned}
|\langle M^{-1}(t).P^{-1}(t)x, P^{-1}(t)x\rangle| &= |\langle P^{-T}M^{-1}P^{-1}x, x\rangle| = |\langle (U-V)x, x\rangle| \\
&|\langle Ux, x\rangle| + |\langle Vx, x\rangle| \\
&= |x^T U x| + |x^T V x| \\
&\leq x^T(|U|_5 + |V|_5) I x \\
&= x^T x (|U|_5 + |V|_5) \\
&= \|x\|^2 (|U|_5 + |V|_5) \quad \text{lemma 5:9.}
\end{aligned}$$

since U,V are symmetric.

$$\begin{aligned}
&= (|U|_5 + |V|_5) \quad \text{since x is unit.} \\
&\leq (|U| + |V|)c \text{ by equiv of norms.} \\
&\leq 2cL^2 \quad \text{by lemma6:6 above.}
\end{aligned}$$

If $\lambda(s)$ is the largest eigenvalue of $M(s)$ then $1/\lambda(s)$ is the smallest eigenvalue of $M^{-1}(s)$ for $s \geq 0$. M is positive definite so all the eigenvalues are positive. So $|M|_2 = \lambda(s)$.

Hence $1/\lambda(s) \, |P^{-1}(t)x|^2 \leq (P^{-1}x)^T M^{-1}(t)(P^{-1}x)$

$= |\langle M^{-1}(t)A^{-1}(t)x, A^{-1}(t)x\rangle| \leqslant 2cL^2$ by previous.

So $\|P^{-1}(t)x\|^2 \leqslant (2cL^2)\lambda(s) = (2cL^2)\,|M(t)|_2$

Since x is a unit vector we take the operator norm to obtain: $|P^{-1}(t)|_3 \leqslant (2cL^2)^{\frac{1}{2}}|M(t)|_2$

Hence if $y \in \mathbb{R}^{n-1}$ is another unit vector:

$\|P(t)y\| \geqslant 1/|P^{-1}(t)|_3 \geqslant 1/(2cL^2|M(t)|_2)^{\frac{1}{2}}$

By definition $M(t) \to 0$ as $t \to \infty$, so given $R > 0$ there is t_o st $|M(t)|_2 \leqslant 1/2cL^2R^2$ for $t > t_o$.

So $\|P(t)y\| \geqslant R$ for t t_o

Result follows since the above is true for all unit y.

q.e.d.

We can use this to characterise $\mathcal{Y}_c$.

Proposition6:8. If Y is a normal k-field along a k-line st $\|Y(t)\|$ is bdd above for all t then $Y \in \mathcal{Y}_c$.

Proof: Need to show that $Y(t) = W(t)c$ for some constant c. Consider the normal k-field $Y_r(t) = W_r(t)Y(0)$ for W, W_r as in lemma 6:2. $W(0) = W_r(0) = I$.

Let $Z_r(t) = (Y-Y_r)(t)$ be a normal k-field st $Z_r(0) = (Y-Y_r)(0) = 0$. Now $\|Y(t)\| \leqslant A$ for some A, and all $t > 0$.

then $|Z_r(r)| = |Y_r(r) - Y(r)| \leqslant |Y(r)| + |Y_r(r)| \leqslant A + 0$

Choose $R > 0$, then there is t_R st given a normal k-field Z along c st $Z(0) = 0$ then $|Z'(0)| \leqslant 1/R \; |Z(t)|$ for $t \geqslant t_R$.

Hence there is $r > t_R$ st $|Z_r'(0)| \leqslant 1/R \; |Z_r(r)| \leqslant A/R$.

Let $R \to \infty$ then $0 = \lim_{r\to\infty} |Z_r'(0)| = \lim_{r\to\infty} |(Y_r' - Y')(0)|$

$$= \lim_{r\to\infty} |(Y_r'(0)) - Y'(0)|$$

implies that $Y'(0) = \lim_{r\to\infty} W_r'(0)\, c = W'(0)\, c$

So $Y(0) = W(0)\,c$, $Y'(0) = W'(0)\,c$, and by uniqueness of solution given by initial conditions then $Y(t) = W(t)\,c$

q.e.d.

Proposition 6:9: There is $d^2 > 0$ st if $X \in \mathfrak{X}_c$ for any k-line c then $\|\nabla_{v(t)} X\| \leqslant d^2 \; \|X(t)\|$.

Proof: If $X \in \mathfrak{X}_c$ let Y be its normal k-field, then $Y(t) = W(t)\,c$ for some constant c and W as in lemma 6:3.

$$\|\nabla_{v(t)} X\|^2 = |\nabla_v X|^2 = (X_0' - k_1 X_1)^2 + \left| \begin{bmatrix} X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}' + \begin{bmatrix} k_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad B \quad \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix} \right|^2$$

$$= \quad 0 + |Y' + AX| \text{ say since } X_0' - k_1 X_1 = 0$$

by definition of k-field, where $\nabla_v X \perp v$.

Hence $\|\nabla_v X\| = |Y' + AX| \leqslant |Y'| + |AX|$

A is an (n-1)xn matrix and X a column matrix, so $\|\nabla_v X\| \leqslant |Y'| + |A|_3 |X|$, the operator norm.

$Y'(t) = W'(t)c$ so:

$$\|Y'(t)\| = |W'(t)c| = |W'W^{-1}Wc| \leqslant |W'W^{-1}|_3 |Wc| = |W'W^{-1}|_3 |Y|$$

$$\leqslant |W'W^{-1}|(n-1)\|Y\| \text{ by equivalence of norms}$$

$$(n-1)L^2 \|Y\| \text{ by lemma6:}\underline{6}$$

and L^2 independent of the k-line.

$$\text{So } \|\nabla_v X\| \leqslant (n-1)L^2 \|Y\| + |A|_3 \|X\| \leqslant ((n-1)L^2 + |A|_3)\|X\|$$

$= d^2 \|X\|$ for d independent of the k-line since A is a constant matrix.

q.e.d.

Corollary6:9:1. There is $d_1^2 > 0$ st if $Y \in \mathcal{Y}_c$ is a normal k-field along any k-line c then $\|Y'(t)\| \leqslant d_1^2 \|Y(t)\|$

Proof: Contained in the above.

Note: The above is true for more general k-fields than those of $\mathcal{Y}_c$.

The Anosov properties also require a contracting subspace as $t \to -\infty$. Consider the reverse k-flow φ_t, whose k-lines are those of the original k-flow with opposite orientation.(lemma2:6 et al).

Definition: Given a k-line c of a k-flow let $\mathfrak{X}_{-c}$ be the space of k-fields of the reverse flow along $-c$.

Lemma 6:10. If $X(t)$ is a $\underline{k}$-field of a $\underline{k}$-flow along c then $X^{-}(t)$ is a $\underline{k}$-field of the reverse flow along $(-c)$ where $X^{-}(t) = X(-t)$.

Proof: If X is a $\underline{k}$-field then it is the variation vector field of a variation through $\underline{k}$-lines, Lemma 3:3; say

$$V ; I \times (-\varepsilon, \varepsilon) \longrightarrow M$$
$$(t, u) \longmapsto c_u(t)$$

where c_u is a $\underline{k}$-line of the flow for fixed u.

Reversing this variation to obtain:

$$V^{-} : I \times (-\varepsilon, \varepsilon) \longrightarrow M$$
$$(t,u) \longmapsto c_u(-t)$$

then V^{-} is a variation through $\underline{k}$-lines of the reverse flow, since $c_u(-t)$ is a reverse $\underline{k}$-line of $c_u(t)$. Moreover X with the opposite orientation is the variation vectorfield, ie X^{-}.

q.e.d.

Corollary 6:8:1. If φ_t is a self adjoint $\underline{k}$-flow on M then the reverse $\underline{k}$-flow, ψ_t, is self adjoint.

By above X is a $\underline{k}$-field of φ_t $\quad X^{-}$ is a $\underline{k}$-field of ψ_t

Consider the normal parts of X, X^{-}, say Y, Y^{-} resp.

$Y(t) = Y^{-}(-t)$ and so if (Y,Z) satisfy:

$$\begin{cases} Y' = AY + Z \\ Z' = -CY - A^T Z \end{cases} \quad \text{where} \quad \begin{bmatrix} C & A^T \\ A & I \end{bmatrix} \text{ is symmetric}$$

then (Y^{-}, Z^{-}) satisfies:

$$\begin{cases}(Y^-)' = (-A(-t))\,Y^- + Z^- \\ (Z^-)' = -\,(C(-t))\,Y^- - (-A(-t))^T Z^-\end{cases}$$

and the matrix $\begin{bmatrix} C(-t) & -A(-t)^T \\ -A(-t) & I \end{bmatrix}$ is also symmetric

and hence the reverse k-flow is self adjoint. (Introduction to Section 5).

q.e.d.

Definition: For all $v \in T^1M$ let $E^u_v = \{(x, v, X(0) \oplus \nabla_{v(0)} X) : X \in \mathfrak{X}_{-c} \}$

<u>Proposition6:11</u>. E^u_v is $T\varphi_t$ invariant, $T\varphi_t(E^u_v) = E^u_{\varphi_t(v)}$.

Proof: If Ψ_t is the reverse k-flow then by Prop^{n}6:4

$$T\,\Psi_t(E^u_v) = E^u_{\Psi_t(v)}. \qquad T\Psi_{-t}(E^u_v) = E^u_{\Psi_{-t}(v)}$$

Since Ψ_t is the reverse then $\Psi_t\,\varphi_{-t} = \varphi_t\,\Psi_{-t} = \mathrm{id}$.

Hence $\quad T\varphi_t(E^u_v) = E^u_{\varphi_t(v)}$

q.e.d.

We can now give sufficient conditions for the Anosovity of a k-flow in terms of the behaviour of its k-lines (Variations) and in particular the two families just constructed.

Proposition6:12. A $\underline{k}$-flow on a compact manifold, M, disconjugate for the flow is Anosov if given $\underline{k}$-fields $X \in \mathcal{X}_d$, $X^- \in \mathcal{X}_{-d}$, along any $\underline{k}$-line d of the flow then their normal $\underline{k}$-fields Y, Y^- resp satisfy:

$$\|Y(t)\| \leqslant a\,\|Y(0)\|\,e^{-ct}$$

$$\|Y^-(t)\| \leqslant b\,\|Y^-(0)\|\,e^{-ct}$$

for $t \geqslant 0$ and $a,b,c > 0$ constants independent of the $\underline{k}$-line.

Proof: If $X \in \mathcal{X}_d$ $(X(0) \oplus \nabla_{v(0)}X) \in E^s_v$

By Prop^{n}6:4 $T\varphi_t\,(x,v,X \oplus \nabla_v X)) \in E^s_{v(t)}$

By lemma6:5 $\|X(t)\| \leqslant l^2 \|X(0)\|\,e^{-ct}$ and

by Prop^{n}6:9 $\|\nabla_{v(t)}X\| \leqslant l^2 d^2 \|X(0)\|\,e^{-ct}$ for $t \geqslant 0$.

$$\begin{aligned}
\text{Hence}\quad \|\!|T\varphi_t\,(X \oplus \nabla_v X)\|\!|^2 &= \|X(t)\|^2 + \|\nabla_{v(t)}X\|^2 \\
&\leqslant (1+d^2)l^2a^2\,\|X(0)\|^2\,e^{-2ct} \\
&\leqslant (1+d^2)l^2a^2(\|X(0)\|^2 + \|\nabla_{v(0)}X\|^2)e^{-2ct} \\
&= p^2\,\|\!|X(0) \oplus \nabla_{v(0)}X\|\!|^2\,e^{-2ct}
\end{aligned}$$

for $t > 0$.

Hence one of the four inequalities of Anosov is satisfied.

(ii) By invariance for $t \leqslant 0$:

If $(X \oplus \nabla_v X) \in E^s_v$ then

$$\|\!|(X \oplus \nabla_v X)\|\!| = \|\!|T\varphi_{-t}\,T\varphi_t(X \oplus \nabla_v X)\|\!| \leqslant p\,\|\!|T\varphi_t(X \oplus \nabla_v X)\|\!|\,e^{+ct}$$

So $\|\!|T\varphi_t\,(X \oplus \nabla_v X)\|\!| \geqslant 1/p\,\|\!|X \oplus \nabla_v X\|\!|\,e^{-ct}$.

Hence we have both inequalities for E^s_v.

(iii) If $(X \oplus \nabla_v X) \in E^u_v$ then $X^- \in \mathcal{X}_{-d}$, for the reverse flow. We have

$$|||T\varphi_t(X \oplus \nabla_v X)|||^2 = \|X^-(-t)\|^2 + \|\nabla_{-v(t)} X^-\|^2$$

and hence by the same analysis as above:

$$\leqslant (1+d^2)b^2 l^2 \|X^-(0)\|^2 e^{+2ct} \quad \text{(using } -t\text{)}$$

$$\leqslant (1+d^2)b^2 l^2 (\|X\|^2 + \|\nabla_v X\|^2) e^{\pm 2ct}$$

$$= q^2 |||X \oplus \nabla_v X|||^2 e^{2ct}.$$

Hence $|||T\varphi_t(X\oplus\nabla_v X)||| \leqslant q\, |||X \oplus \nabla_v X|||\, e^{ct}$ for $t \leqslant 0$.

This is the first inequality established for E^u_v and the other is established by (ii).

(iv) This is the inequalities established, we now have to show that $T_v T^1 M = E^s_v \oplus E^u_v \oplus Z_v$.

$\dim E^s = \dim E^u = n-1$ and $\dim Z = 1$, so if E^s, E^u, Z are mutually exclusive then the splitting is valid, since $\dim T_v T^1 M = 2n-1$.

If $(X \oplus \nabla_v X) \in Z_v$ then $X(0) = cv \oplus ck_1 N_1$.

So $T\varphi_t(X \oplus \nabla_v X) = (X(t) \oplus \nabla_{v(t)} X)$ is a <u>k</u>-field with initial conditions $(cv, ck_1 N_1)$, and by lemma3:6:2,

$$\begin{cases} X(t) = cv(t) \\ \nabla_{v(t)} X = ck_1 N_1(t) \quad \text{for all } t. \end{cases}$$

So $|||X(t) \oplus \nabla_{v(t)} X||| = |c|\,(1 + k_1^2)^{\frac{1}{2}}$ ie a constant for $t \geqslant 0$

but if $(X \oplus \nabla_v X) \in E^s_v$ $\quad |||X(t) \oplus \nabla_{v(t)} X||| \to 0$ as $t \to \infty$.

$(X \oplus \nabla_v X) \in E^u_v$ $\quad |||X(t) \oplus \nabla_{v(t)} X||| \to \infty$ as $t \to \infty$.

Hence we have mutual exclusivity.

q.e.d.

Corollary6:12:1. A reversible $\underline{k}$-flow is Anosov if given any $\underline{k}$-field $X \in \mathfrak{X}_c$ on any $\underline{k}$-line c, with normal part Y then

$$\|Y(t)\| \leqslant a \|Y(0)\| e^{-kt} \quad \text{for } t \geqslant 0, k > 0.$$

Proof: $\mathfrak{X}_{-c} = \mathfrak{X}_d$ where $d(t) = c(-t)$ since the reverse $\underline{k}$-lines are also $\underline{k}$-lines of the flow. Hence the second set of conditions are satisfied.

q.e.d.

Example: For the geodesic flow we need only check that any normal Jacobi field satisfies an exponential condition, which classically is guaranteed by working on negative curvature manifolds.

Section 7: On Manifolds of Negative curvature.

It is known that on a manifold of strict negative curvature that the geodesic flow is Anosov. (Anosov [2]) By studying the $\underline{k}$-flows we have introduced curvature terms due to the flow lines, via the k_i and the $N_{\underline{k}}$; we should like to know if we can make similar statements by assigning to each flow a number K st if $K < 0$ then the flow is Anosov, this number depending on $\underline{k}$, $N_{\underline{k}}$ and the curvature of the manifold. To allow us to put simple bounds on the curvature we now work on compact manifolds.

From the last section we saw that a $\underline{k}$-flow is Anosov if the normal $\underline{k}$-fields $Y \in \mathcal{Y}_c$ satisfy:

$$\|Y(t)\| \leqslant C \|Y(0)\| e^{-ct}$$

The invariance of $\mathcal{Y}_c$ thus implies that:

$$\forall s, \ \|Y(t)\| \leqslant C \|Y(s)\| e^{-(t-s)} \quad \text{for } t > s.$$

In particular this is true when $\frac{d^2}{dt^2}(\|Y\|) \geqslant c_1(\|Y\|) > 0$.

Hence we begin by studying:

Lemma 7:1. If Y is a normal $\underline{k}$-field along a $\underline{k}$-line then $\frac{1}{2} d^2/dt^2 (\|Y\|^2) = \|\nabla_v Y\|^2 + k_1(DY').Y + (-R - 2M + k_1 E)Y.Y$

Proof: Y is a vectorfield along the $\underline{k}$-line with matrix representation $(0, Y_1, \dots, Y_{n-1})^T$

Using lemma0:1

$$d^2/dt^2 (\|Y\|) = d^2/dt^2 (\langle Y, Y \rangle) = 2\langle \nabla_v^2 Y, Y \rangle + 2\|\nabla_v Y\|^2$$

$$\nabla_v^2 Y \equiv \begin{bmatrix} 0 \\ Y_1 \\ \vdots \\ Y_{n-1} \end{bmatrix}'' + 2 \left[\begin{array}{cccc|c} 0 & -k_1 & 0 & & \\ k_1 & 0 & -k_2 & & \\ 0 & k_2 & 0 & \ddots & 0 \\ & & & 0 & \\ \hline & & 0 & & 0 \end{array}\right] \begin{bmatrix} 0 \\ Y_1 \\ \vdots \\ Y_{n-1} \end{bmatrix}' + \begin{bmatrix} -k_1^2 & 0 & k_1k_2 & 0 & \cdots & 0 \\ 0 & & & & & \\ k_1k_2 & & & C & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \begin{bmatrix} 0 \\ Y_1 \\ \vdots \\ Y_{n-1} \end{bmatrix}$$

by Cor4:1:1

$$\equiv \begin{bmatrix} 0 \\ Y_1 \\ \cdot \\ \cdot \\ Y_{n-1} \end{bmatrix}'' + 2\begin{bmatrix} 0, & -k_1 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & & & \\ \vdots & & B & & \\ 0 & & & & \end{bmatrix}\begin{bmatrix} 0 \\ Y_1 \\ \cdot \\ \cdot \\ Y_{n-1} \end{bmatrix}' + \begin{bmatrix} -k_1^2 & 0 & k_1k_2 & \cdots & 0 \\ 0 & & & & \\ k_1k_2 & & & & \\ 0 & & C & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}\begin{bmatrix} 0 \\ Y_1 \\ \cdot \\ \cdot \\ Y_{n-1} \end{bmatrix}$$

where $B = \left[\begin{array}{cccccc|c} 0 & -k_1 & 0 & & & & \\ k_1 & 0 & -k_2 & & & & \\ 0 & k_2 & 0 & \ddots & & & \underset{\sim}{0} \\ & & & \ddots & 0 & -k_r & \\ & & & & k_r & 0 & \\ \hline & & \underset{\sim}{0} & & & & \underset{\sim}{0} \end{array}\right]$ $C = \left[\begin{array}{cccccc|c} -k_1^2-k_2^2 & 0 & k_2k_3 & & & & \\ 0 & -k_2^2-k_3^2 & 0 & & & & \\ k_2k_3 & 0 & -k_3^2-k_4^2 & & & & \underset{\sim}{0} \\ & & & \ddots & -k_r^2-k_{r-1}^2 & 0 & \\ & & & & 0 & -k_r^2 & \\ \hline & & \underset{\sim}{0} & & & & \underset{\sim}{0} \end{array}\right]$

$C = T - 2M$ {see Prop^n 4:7 and the notation following Prop5:1}

Now $X.Y = \langle X,Y\rangle$ lemma 4:2

So $\langle\nabla_v^2 Y, Y\rangle = (Y'' + 2BY' + (T-M)Y).Y$

and Y satisfies $Y'' + 2BY' - k_1DY' + (R + T - k_1E)Y = 0$

So $Y'' + 2BY' + (T-M)Y = k_1DY' + (-R + k_1E - 2M)Y$

q.e.d.

<u>Proposition7:2.</u> There is L_0 st if Y is a normal <u>k</u>-field st $Y(t) \neq 0$ on $[0,s]$ then:

$$\frac{d^2}{dt^2}(\|Y\|) > L_0\|Y\| \qquad \text{for } t \in [0,s]$$

Proof: $d/dt\ \|Y\| = d/dt \langle Y,Y\rangle^{\frac{1}{2}}$

$$= \tfrac{1}{2}\langle Y,Y\rangle^{-\frac{1}{2}}\ d/dt\langle Y,Y\rangle$$

$$= \frac{\langle\nabla_v Y, Y\rangle}{\|Y\|}$$

So $(d/dt\|Y\|)^2 = \dfrac{\langle\nabla_v Y, Y\rangle^2}{\|Y\|^2} \leqslant \dfrac{\|\nabla_v Y\|^2\|Y\|^2}{\|Y\|^2} = \|\nabla_v Y\|^2$

Hence $\left(\frac{d^2}{dt^2}\|Y\|\right)\|Y\| = \frac{1}{2}\left(\frac{d^2}{dt^2}\|Y\|^2\right) - \left(\frac{d}{dt}\|Y\|\right)^2$

$$\geqslant k_1 DY'.Y + (-R + k_1E - 2M)\,Y.Y \qquad \text{lemma7:1}$$

Now $X.Y = \quad X^iY^i = Y^TX \in \mathbb{R}$. So

$$|DY'.Y| = |Y^T(D'Y)| \leqslant |DY'|_1 \, |Y^T|_1$$

$$\leqslant |D|_1 \, |Y'|_1 \, |Y|_1$$

$$\leqslant |D|_1 \, (n-1)^2 \|Y'\| \, \|Y\| \qquad \text{norm equivalence}$$

$$\leqslant (n-1)^2 \, |D|_1 \, L_1^2 \|Y\|^2 \qquad \text{Cor6:9:1}$$

$$\leqslant (n-1)^2 \, DL_1^2 \|Y\|^2 \quad \text{for } D \text{ as Cor5:10:1}$$

Similarly $|EY.Y| = |Y^TEY| \leqslant Y^T|\tfrac{1}{2}(E+E^T)|_4 \, I.\,y \qquad$ Cor5:10:1

$$\leqslant E\|Y\|^2 \qquad \text{for } E \text{ as Cor5:10:1}$$

$$|2MY.Y| \leqslant 2k_1^2 \|Y\|^2 \qquad \text{by definition of } M$$

We are in a manifold of negative curvature so by compactness there is $K_2^2 \geqslant K_1^2 \geqslant 0$ st curvature lies in $[-K_2^2, -K_1^2]$ Hence RY.Y satisfies:

$$-K_2^2\|Y\|^2 \leqslant RY.Y \leqslant -K_1^2\|Y\|^2 \qquad \text{Cor4:3:3}$$

Hence

$$\left(\frac{d^2}{dt^2}\|Y\|\right)\cancel{\|Y\|} \geqslant (K_1^2 - (n-1)^2L_1^2D - 2k_1^2 - k_1E)\|Y\|^{\cancel{2}}$$

$$\frac{d^2}{dt^2}\|Y\| \geqslant (K_1^2 - K_3^2)\|Y\| \text{ say.}$$

q.e.d.

<u>Corollary7:2:1</u>. If $Y \in \mathcal{Y}_c$ then $\frac{d^2}{dt^2}\|Y\| \geqslant L_0 \|Y(t)\|$ for $t \geqslant 0$.

Proof: $Y(t) \neq 0$ for $t \geqslant 0$ by construction.

q.e.d.

In the above inequality L_0 may well be negative, hence to assign a number K to each $\underline{k}$-flow st if $K < 0$ the flow is Anosov then we have two considerations:-

(i) M is disconjugate for the flow.

(ii) $L_0 > 0$ in the above inequality.

Ideally this should be constructed from knowledge of the manifold M, $\underline{k}$ and the $\underline{k}$-section, $N_{\underline{k}}$, without a detailed inspection of how the flow lines are related.

To date we have used the following estimates.

(i) The curvature of M lies in $[-K_2^2, -K_1^2]$

(ii) $\underline{k} = (k_1, \ldots, k_r)$ $\quad B = \left[\begin{array}{cccc|c} 0 & -k_2 & & & \\ k_2 & 0 & & & \\ & & \ddots & & 0 \\ & & 0 & -k_{r+1} & \\ & & k_r & 0 & \\ \hline & & 0 & & 0 \end{array}\right]$

(iii) the flow has disconjugacy number $K_4^2 < \infty$.

(iv) We have a map $N_1 : T^1M \to T^1M$ st the map KoTN gives the matrices $[D_{ij}], [E_{ij}]$. Hence we can find the estimates D, E we used above without direct reference to the $\underline{k}$-lines themselves, Cor5:10:1.

Definition: Let $K_3^2 = (n-1)^2 D L_1^2 + 2k_1^2 + k_1 E$ where

$$L_1^2 = (n-1)^{5/2} K_1 + |B| + \tfrac{1}{2} k_1 D.$$

By the above we can thus estimate K_3^2 by simply (?) considering $\underline{k}$ and $N_{\underline{k}}$. This may not be the closest estimate for L_0, for instance we could replace the constant L_0 by $L_0(t) = (n-1)^2 |D(t)|_1 ((n-1) K_2 + \tfrac{1}{2} k_1 (n-1) |D(t)| + (n-1)B) - 2k_1^2 - \tfrac{1}{2} k_1^2 |E(t) + E^T(t)|_4$, and letting $L_0 = \max_t L_0(t)$, taken over the $\underline{k}$-lines of the flow, and completions of $N_{\underline{k}}$. However we are attempting to obtain estimates based only on the manifold, $\underline{k}$ and the $\underline{k}$-section, so while recognising the possible refinement we shall keep to our D, E estimate.

Definition: Given a $\underline{k}$-flow on a manifold M, compact and with negative curvature let $K = \max[K_4^2 - K_1^2, K_3^2 - K_1^2]$ be the <u>Anosov number</u>.

Examples. (i) The geodesic flow on manifolds of negative curvature has $K_4^2 = 0$, $k_i = D = E = 0$ and so $K = \max(-K_1^2, -K_1^2)$ $= -K_1^2$ and we obtain the usual curvature condition $K < 0$.

(ii) For a $\underline{k} = (k)$ flow on a compact oriented surface of constant negative curvature $-K_1^2$, $K_3^2 = 0 + 2k^2 + k^2 = 3k^2$, so $K = \max[2k^2 - K_1^2, 3k^2 - K_1^2] = 3k^2 - K_1^2$. In justification of the remark that we may not have chosen the finest estimate for L_0^2 if the analysis is carried out $\underline{k}$-line by $\underline{k}$-line then we can show that $K_1^2 - k^2 < 0$ is sufficient to guarantee $L_0 > 0$, but we should still like to remain with the D,E estimate.

<u>Theorem 7:3</u>. A $\underline{k}$-flow on a compact manifold of negative curvature is Anosov if the Anosov number $K < 0$.

Proof: If $K < 0$ then the manifold is disconjugate for the flow since $K_4^2 - K_1^2 < 0$ ..... Prop^{n}5:11; also $L_0 = K_1^2 - K_3^2 > 0$ so the constant of Prop^{n}7:2 is positive. We have to show that if $Y \in \mathcal{Y}_c$ then it satisfies an exponential condition.

If $Y \in \mathcal{Y}_c$ then $Y(t) = W(t)c = \lim_{s \to \infty} W_s(t)c = \lim_{s \to \infty} Y_s(t)$ say.

Put $r_s(t) = \|Y_s(t)\| = |Y_s(t)|$ $\qquad r(t) = \|Y(t)\|$

For fixed t $\lim_{s \to \infty} |Y_s(t) - Y(t)| = 0$.

Since $0 \leq \big||Y(t)| - |Y_s(t)|\big| \leq |Y_s(t) - Y(t)|$

then $\lim_{s \to \infty} \|Y_s(t)\| = \|Y(t)\|$ ie $r_s(t) \longrightarrow r(t)$.

By Prop^{n}7:2 $\quad \dfrac{d^2 r_s}{dt^2} \geq L_0\, r_s(t)$ $\quad$ for positive L_0, $r_s(0) > 0$

and $r_s(s) = 0$

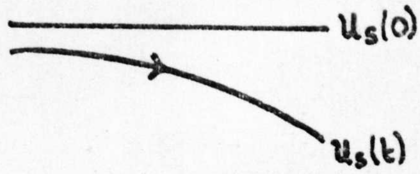
<u>Claim</u> $\quad r_s(t) \leq r_s(0)\, \dfrac{\cosh\sqrt{L_0}\,(s-t)}{\cosh\sqrt{L_0}\, s}$

Proof is an expansion of a sketch in Avez(151)

Consider $u_s(t) = r_s(t) - r_s(0)\dfrac{\cosh\sqrt{L_0}\,(s-t)}{\cosh\sqrt{L_0}\, s}$

$u_s(0) = 0$, $u_s(s) = -r_s(0) < 0$, $\quad \ddot{u}_s(t) = \ddot{r}_s(t) - L_0 r_s(0)\dfrac{\cosh\sqrt{L_0}\,(s-t)}{\cosh\sqrt{L_0}\, s}$

so $\ddot{u}_s(t) \geq 0$ by hypothesis.
$u_s(t)$ is thus a concave function
and so $u_s(t) < 0$ for all $t > 0$.



Hence $\quad \|Y_s(t)\| \leq \|Y_s(0)\|\dfrac{\cosh\sqrt{L_0}\,(s-t)}{\cosh\sqrt{L_0}\, s}$

$Y(0) = Y_s(0)$ for all s since $W_s(0) = W(0) = I$, and so:-

$$\|Y(t)\| = \lim_{s\to\infty}\|Y_s(t)\| \leq \|Y(0)\| \lim_{s\,\infty} \frac{\cosh\sqrt{L_0}\,(s-t)}{\cosh\sqrt{L_0}\, s}$$

$$= \|Y(0)\| \lim_{s\to\infty}\left\{\frac{\cosh\sqrt{L_0}s\,\cosh\sqrt{L_0}t}{\cosh\sqrt{L_0}s} + \frac{\sinh\sqrt{L_0}s\,\sinh\sqrt{L_0}t}{\cosh\sqrt{L_0}s}\right\}$$

$$= \|Y(0)\| \lim_{s\,\infty}\left\{\cosh\sqrt{L_0}t + \sinh\sqrt{L_0}t\,\text{than}\,L_0 s\right\}$$

$$= \|Y(0)\|\{\cosh-\sqrt{L_0}t + \sinh-\sqrt{L_0}\,t\} \text{ since } \text{than}x \to 1 \text{ as } x\to\infty$$

$$= \tfrac{1}{2}\|Y(0)\|\{e^{-L_0 t} + e^{L_0 t} + e^{-L_0 t} - e^{L_0 t}\}$$

$$= \|Y(0)\|\, e^{-L_0 t} \quad \text{for } t > 0$$

L_0 is independent of the <u>k</u>-line, so we have the exponential condition for $Y \in \mathcal{Y}_c$.

Consider the reverse flow:

Lemma7:4. (i) If a $\underline{k}$-flow is disconjugate on M so is the reverse flow.

(ii) If D,E are the variation constants for a $\underline{k}$-flow then they are the constants for the reverse flow.

Proof: If the reverse $\underline{k}$-flow were not disconjugate then there is a $\underline{k}$-line d and a normal $\underline{k}$-field Y along d with $Y(a) = Y(b) = 0$ for $a \neq b$. Then $Y^-(t) = Y(-t)$ is a normal $\underline{k}$-field along $c^-(t)$ of the disconjugate flow st $c^-(-a) = c^-(-b) = 0$, contradiction.

(ii) If $N_{\underline{k}}$ is the $\underline{k}$-section of the flow then $N'_{\underline{k}} = (\ldots,(-1)^{i+1}N_i(-v),\ldots.)$ is the reverse $\underline{k}$-section, lemma2:6.

Hence $N' : T^1M \longrightarrow T^1M : (x,w) \longmapsto (x,N_1(-w))$ is the analogue of N_1, and

$TN' : TT^1M \longrightarrow TT^1M : (x,w,e,f) \longmapsto (x,N(-w),e,D_xN(e)-D_{-w}N(f))$

So if $(v,\ldots,N_i,\ldots,E_j,\ldots)$ is a completion for $N_{\underline{k}}$ then $(-v,\ldots,(-1)^{i+1}N_i(-v),\ldots,(-1)^{r+j+1}(E_j(-v)),\ldots.)$ is a completion for $N'_{\underline{k}}$, say F'_j.

$$\begin{aligned} D'_{ij}(-v) &= \langle K \circ TN'(x,-v,0 \oplus F'_j),F'_i\rangle && \text{by definition} \\ &= \langle (-1)^{i+j+1} \quad -D_vN(F_j),F_i\rangle && \text{by expansion \& linearity} \\ &= (-1)^{i+j}D_{ij}(v) \end{aligned}$$

$$\text{So } |D'(-v)|_1^2 = \sum_{i,j=1}^{n-1} |D'_{ij}| = \sum_{i,j=1}^{n-1} |D_{ij}| = |D(v)|_1^2$$

So $D = D'$ by definition in Cor5:10:1

Similarly expanding E and using linearity properties shows $E'_{ij} = (-1)^{i+j}E_{ij}$ and so $E = E'$

q.e.d.

(Completion of proof of Theorem7:3)

So with the reverse $\underline{k}$-flow we have the same estimates as for the original flow, ie same disconjugacy number, same D,E and hence the same Anosov number. Applying the analysis of the first part to the reverse normal $\underline{k}$-fields show they have the same exponential condition as the original flow.

Hence by Prop^{n}6:12 the flow, and its reverse flow is Anosov.

q.e.d.

Corollary7:3:1. If a $\underline{k}$-flow has Anosov number <0 then the reverse flow is also Anosov.

Proof: shown above.

Remarks: (i) As expected in the special case of the geodesic flow we obtain Anosov's result that a geodesic flow on a manifold of strict negative curvature is Anosov.

(ii) A $\underline{k}=(k)$-flow on an oriented surface of constant negative curvature $-K^2$ is Anosov if $K^2 < 3k^2$. (Actually if $K^2 < 2k^2$ if we consider the remark preceding the definition of Anosov number)

Section 8: Necessary and sufficient conditions for the Anosovity of self adjoint k-flows.

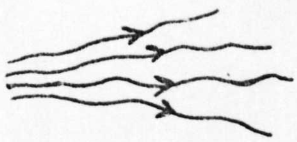
The previous section gave sufficient conditions in terms of quantities directly measurable from the k-section. We should now like to give some necessary conditions to complement this work and generalise the work of Eberlein ([7]), on the geodesic case.

As previously we shall use the normal k-fields to reflect the variation properties, and to measure the growth of these k-fields we use the following two functions.

(i) Since we built up the $\mathcal{Y}_c$ from normal k-fields satisfying $Y(0) = 0$, $Y'(0) = I$ consider:

$$g: \mathbb{R}^+ \to \mathbb{R}^+ : s \longmapsto \inf\left\{\|Y(s)\| : \begin{array}{l} Y \text{ is a normal } k\text{-field along} \\ \text{some } k\text{-line st } Y(0)=0, \|Y'(0)\| = 1. \end{array}\right\}$$

This function measures the 'spread' of the k-lines through a point $m \in M$, since adding a tangential component gives variations of the form:



(ii) $$\varphi ; \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : s \longrightarrow \left\{\begin{array}{l} \sup \|Y(s)\| : Y \in \mathcal{Y}_c \text{ st } \|Y(0)\| = 1 \\ \text{over all } k\text{-lines } c. \end{array}\right\}$$

In some of the following proofs we shall consider sequences of normal k-fields along k-lines, making use of the following properties of compact manifolds:

Eberlein ([7]), Given compact M and sequences $\{v_n\} \in TM$ and $\{\xi_n\} \in T_{v_n}TM$ st $\|v_n\|, \|\xi_n\|$ are uniformly bdd above then passing to a subsequence $v_n \to v$, $\xi_n \to \xi \in T_v TM$.

Définition: On a compact manifold M, disconjugate for a $\underline{k}$-flow, given a sequence of normal $\underline{k}$-fields, $\{Y_n\}$, on $\underline{k}$ lines c_n, they converge to a normal $\underline{k}$-field Y along a $\underline{k}$-line c if $\dot{c}_n(0)\to\dot{c}(0)$, $Y_n(0)\to Y(0)$, $\dot{Y}_n(0)\to\dot{Y}(0)$.

This is well defined since specifying $\dot{c}(0)$ determines the $\underline{k}$-line uniquely and by disconjugacy $Y(0),\dot{Y}(0)$ the normal $\underline{k}$-field.

Lemma 8:1. (Compact Property) Let $\{Y_n\}$ be a sequence of normal $\underline{k}$-fields along $\underline{k}$-lines $\{c_n\}$ on a compact manifold disconjugate for the flow; then if $\|Y_n(0)\|$ $\|\dot{Y}_n(0)\|$ are uniformly bdd above for all n, then in subsequence $\{Y_n\}$ tends to a normal $\underline{k}$-field Y along a $\underline{k}$-line c. Moreover if $u_n\to u<\infty$, then $Y_n(u_n)\to Y(u)$, $\dot{Y}_n(u_n)\to\dot{Y}(u)$.

Proof: Set $\mathfrak{Y}_n = (c_n(0),\dot{c}_n(0),Y_n(0),\dot{Y}_n(0)) \in TTM$. For a $\underline{k}$-line $\|\dot{c}_n(0)\|\equiv 1$, so Eberlein's result gives the first part.

The second part follows by reparameterising c_n to give $\gamma_n(s) = c_n(u_n+s)$ and considering $Z_n(s) = Y_n(u_n+s)$ along it. Taking the subsequence $Y_n\to Y$, by uniqueness of normal $\underline{k}$-fields given initial conditions says $Z_n\to Z$ which is Y reparameterised.

q.e.d.

Remark: As Eberlein we could work on more general manifolds than compact, ie those whose Universal cover is compact homogeneous, where there is a compact $B^*\subseteq M^*$ st M^* is the union of the translates of B^* under the isometries of

M^*. All the metric properties of M^*, hence by projection M, can be pulled back into the compact set B^* and the above analysis performed there, giving the estimates for D,E as before and hence an Anosov number. Hence we could prove our previous theorems on this more general M; the proofs however being more complicated and only technical details being added, rather than new ideas. Hence we work on compact manifolds.

Eberlein shows that the function g satisfies:

Lemma 8:2. $g(s) > 0$ and lower semi-cts for $s > 0$

Proof: g is the inf of a set of positive semi-cts fncs $\{\|Y(s)\|$ st $Y(0)=0,\ \|\dot{Y}(0)\| = 1.\}$ Given $\alpha > 0$ then

$\{s: g(s) < \alpha\}$ is open in $(0, \infty)$, so given s_o g is semi-cts at s_o by choosing $\alpha = g(s_o)$

To show that $g(s) > 0$ we use the fact that there is a Y st $g(s) = Y(s)$ and by disconjugacy $Y(s) \neq 0$ for $s > 0$.

Since g is an inf there is a family of normal k-fields $\{Y_n\}$ on k-lines c_n st $Y_n(0) = 0$, $\|\dot{Y}_n(0)\| = 1$ and $\|Y_n(s)\| \to g(s)$. By the compact property $Y_n \to Y$ along c, since $\|Y_n(0)\| = 0$, $\|\dot{Y}_n(0)\| = 1$. By continuity

$$Y(0) = \lim_{n\to\infty} Y_n(0) = 0,\ \|\dot{Y}(0)\| = \lim_{n\to\infty} \|\dot{Y}_n(0)\| = 1 \text{ and } 0 \neq \|Y(s)\| = g(s).$$

q.e.d.

Proposition 8:3. If M is a compact manifold disconjugate for an Anosov $\underline{k}$-flow then there are no normal $\underline{k}$-fields, non-trivial along any $\underline{k}$-line st $\|Y(t)\|$ is bdd above for all t.

Proof: If $Y(t)$ is a normal $\underline{k}$-field st $\|Y(t)\| < A$ for all t and some constant A then let X be a $\underline{k}$-field along the same $\underline{k}$-line, c, with Y as normal part(eg Solve $\dot{X}_0 = k_1 X_1$ st $X_0(0) = 0$).

$$\mathfrak{J}(t) = (X(t) \oplus \nabla_{v(t)} X) \in T_v T^1 M$$

Anosov implies that $T_v T^1 M = E_v^s \oplus E_v^u \oplus Z_v$ so let.

$$= \mathfrak{J}_1 + \mathfrak{J}_2 + \mathfrak{J}_3$$

$\mathfrak{J}_1(t)$, $\mathfrak{J}_2(t) \neq 0$ since if so $\Rightarrow \mathfrak{J}_3 = (Bv(t) \oplus k_1 B N_1(t))$ for some constant B, and $X(t) \equiv (B,0,\dots,0)^T$ ie $Y \equiv 0$.

So there is t_2 st $\mathfrak{J}_2(t_2) \neq 0$.By the Anosov properties $\|\|T\varphi_t(\mathfrak{J}_2(t_2))\|\|^2 \geq K\|\|\mathfrak{J}_2(t_2)\|\|^2 e^{2ct}$ for $t > t_2$.

ie $\|X(t)\|^2 + \|\nabla_v X\|^2 \geq K\|\mathfrak{J}_2(t_2)\| e^{2ct}$.

By Corollary6:9:1 (remark) $\|\nabla_v X\| \leq K^2 \|X(t)\|$ for $t > 0$

If Y is bdd above for all t then $Y \in \mathcal{Y}_c$ by Prop6:8 $Y \in \mathcal{Y}_c$.

So $\|X(t)\|^2 \geq L/_{(1+K^4)} e^{2ct}$ for $t > 0$.

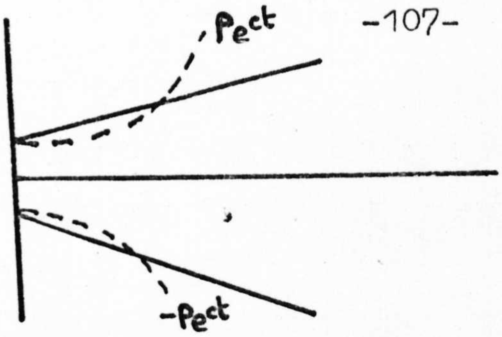
$|X_0(t)|^2 + \|Y(t)\|^2 \geq P^2 e^{2ct}$ implies $|X_0(t)| \geq P^2 e^{2ct} - A^2 \geq R^2 e^{2ct}$.

So $|X_0(t)| \geq Re^{ct}$ for $t \geq 0$.

X_0 is cts so $|X_0(t)| \geq Pe^{ct}$ or $|X_0(t)| < -P e^{ct}$

By the equation for a $\underline{k}$-field $|\dot{X}_0(t)| = k_1 |Y_1(t)| \leq k_1 |Y(t)| \leq k_1 A$

So X_0 is a cts function st $X_0(t) \leq Re^{ct}$ but $\dot{X}_0(t) < Q$ say which is a contradiction by looking at the function $g(t)$ st $g(0) = X_0(0)$, $\dot{g}(0) = Q$ for then $|X_0(t)| \leq g(t)$



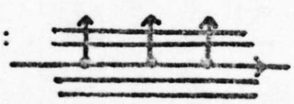
But there is T st $Re^{ct} > g(t) > |X_0(t)|$ contradiction.

If $\xi_1(t_1) \neq 0$ consider the reverse flow which is Anosov and apply the above to get the contradiction.

q.e.d.

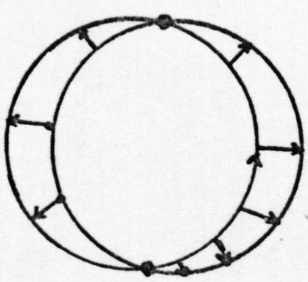
Intuitively this says that if we have an Anosov k-flow then there are no variations through k-lines that stay close together, they must spread out.

Corollary8:3:1 The geodesic flow on $\mathbb{R}^n$ is not Anosov.

Proof:  is a variation with $Y(t) \equiv 1$

Corollary8:3:2. Any k-flow on $\mathbb{R}^2$ is not Anosov.

k-lines are circles with radius $1/k$, so consider



with variation

q.e.d.

To prove the converse of the theorem we need:-

Lemma8:4. If M admits no non-zero normal k-fields Y along any k-line st $\|Y(t)\|$ bdd above for all t, then there is $A > 0$ st if Y is a normal k-field with $Y(0) = 0$ then $\|Y(t)\| \geq A \|Y(s)\|$ for $t \geq s \geq 1$.

Proof: Again this is an adaptation of a result of Eberlein, using our estimates for k-flows.

Suppose false then there is for each $n \in Z^+$ a normal k-field Y_n along a k-line c_n st $Y_n(0) = 0$ $\|\dot{Y}_n(0)\| = 1$, multiplying by a constant if necessary, and numbers $t_n > s_n > 1$ st $\|Y_n(t_n)\| < 1/n \, \|Y_n(s_n)\|$

On $[0, t_n]$ choose $\{u_n\}$ st $\|Y_n(u_n)\| \geq \|Y_n(s)\|$ for $0 \leq s \leq t_n$ ie choose $\{u_n\}$ to maximise $\|Y_n\|$.

Claim $u_n > \delta > 0$.. The proof is as Eberlein, by assuming not and producing a subsequence of normal k-fields $\{Y_n\}$ st $Y(0) = \lim_{n\to\infty} Y_n(u_n) = 0$ by ctuity, so $\lim_{n\to\infty} \|Y_n(u_n)\| = 0$.

Hence since $t_n > 0$ $\|Y_n(u_n)\| \geq \|Y_n(1)\| \geq g(1) > 0$ by lemma 2 contradicting since $\lim \|Y_n(u_n)\| = g(1)$.

(b) For each n consider Y_n on c_n and reparameterise by $\gamma_n(t) = c_n(t + u_n)$. $Z_n(t) = \dfrac{Y_n(u_n + t)}{\|Y_n(u_n)\|}$ since $Y_n(u_n) \neq 0$ by disconjugacy.

then Z_n is a normal k-field st:

(i) $\|Z_n(0)\| = 1$. (ii) $\|Z_n(s)\| \leq 1$ for $-u_n \leq s \leq t_n - u_n$

(iii) $Z_n(-u_n) = 0$. (iv) $\|Z_n(t_n - u_n)\| \leq 1/n$ by hypothesis.

(v) $\|\dot{Z}_n(0)\| = \dfrac{\|\dot{Y}_n(0)\|}{\|Y_n(u_n)\|}$

If W_0 is the self conjugate solution of the matrix equation st $W_0(0) = 0$, $\dot{W}_0(0) = I$ then $Y_n(t) = W_0(t)c$. for some constant c. By lemma6:6 $|\dot{W}_0(t)W_0^{-1}(t)| \leq L^2$

So $\|\dot{Y}_n(t)\| = |\dot{W}_0 W_0^{-1} W_0 c| \leq |\dot{W}_0 W_0^{-1}| \, 3 \|Y_n(t)\| \leq P\|Y_n(t)\|$

So $\|\dot{Z}_n(0)\| \leq P$ and hence $\{\|Z_n(0)\|\}, \{\|\dot{Z}_n(0)\|\}$ are uniformly bdd above.

Choosing a subsequence let $\{Z_n\}$ tend to Z along c. By continuity $\|Z(0)\| = 1$, and so $Z \neq 0$, and subject to the hypothesis that Z is not bdd above for all t.

The remainder of this proof is undiluted Eberlein using lemma 6: 7. We include it for completeness.

There are four cases to consider:-

(i) $t_n - u_n$ and u_n contain bdd subsequences.

Let $t_n - u_n \to t$, and $u_n \to u \geqslant \delta > 0$

By ctuity $Z(t) = 0$ by property (iv) above

$Z(-u_n) = 0$ " " (iii) above

But $-u_n < 0$ and $t \geqslant 0$ so by disconjugacy $Z \equiv 0$ contradiction.

(ii) $u_n \to \infty$ and $t_n - u_n$ contains a bdd subsequence.

Let $t_n - u_n \to t$. By ctuity $Z(t) = 0$ by (iv)

By ctuity and property (ii) $\|Z_n(s)\| \leqslant 1$ for $s \leqslant t$ 1

But Z is a $\underline{k}$-field st $Z(0) = 0$ and $Z \neq 0$ so $Z'(t) \geqslant 0$

Hence by lemma6:7 $\|Z(s)\| \to \infty$ as $s \to -\infty$. contradiction.

(iii) $t_n - u_n \to \infty$ and $u_n \to u < \infty$. By ctuity $Z(u) = 0$.

$\|Z_n(t_n - u_n)\| < 1/n$ by property(iv) implies $\|Z(s)\| \to 0$ as $s \to \infty$. Hence $Z(u) = 0$ and $Z \neq 0$, so same contradiction as above.

(iv) $(t_n - u_n) \to \infty, u_n \to \infty$.

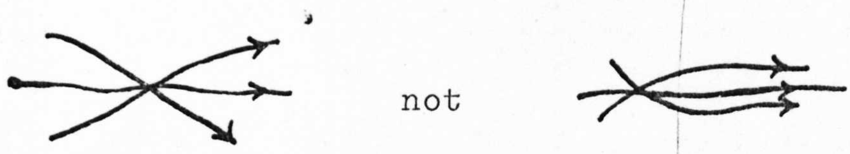
$\|Z_n(s)\| \leqslant 1$ for $-u_n \leqslant s \leqslant t_n - u_n$

$\therefore$ by ctuity $\|Z(s)\| \leqslant 1$ for $-\infty < s < \infty$, contradicting unbdd hypothesis of lemma.

q.e.d.

<u>Corollary8:4:1</u>. If M is a compact manifold disconjugate for a $\underline{k}$-flow, Anosov, then there is $A > 0$ st if Y is a normal $\underline{k}$-field with $Y(0) = 0$ then $\|Y(t)\| \geqslant A \|Y(s)\|$ for all $t \geqslant s \geqslant 1$.

We can interpret this geometrically by saying that it implies that all k-lines passing through a single point diverge exponentially;



This is not as strong from a sufficiency point of view as the condition of lemma8:4 since the geodesic flow on $\mathbb{R}^n$, non-Anosov, satisfies this for A=1.

We study the sufficiency of the condition that M admits no non-trivial normal k-fields bdd above for all t by considering $\varphi(s)$.

Lemma8:5. $0 \leqslant \varphi(s) \leqslant B$ and for all $s \geqslant 0$, for some constant B.

Proof: If $Y \in \mathcal{Y}_c$ then $Y(t) = W(t)c$ for some constant c

By construction $W(t) = \lim_{r\to\infty} W_r(t)$ where $W_r(0) = I$, $W_r(r) = 0$.

Let $Y_r(t) = W_r(t)c$ where $Y_r(0) = c$, $Y_r(r) = 0$

Consider a reparameterisation of c, $c(r-u) = \sigma(u)$, $Z_r(u) = Y_r(r-u)$. So Z_r is a normal k-field st $Z_r(0) = 0$.

By hypothesis and lemma8:4 $\|Z_r(t)\| \geqslant A\, \|Z_r(s)\|$ for all $t \geqslant s \geqslant 1$.

In particular $\|Y_r(t)\| = \|Z_r(t-r)\| \leqslant \frac{1}{A} \|Z_r(r)\| = \frac{1}{A} \|Y_r(0)\|$

$= \frac{1}{A} \|Y(0)\|$

for $r > r-t > 1$ ie $r > t+1$.

For fixed t let $r \to \infty$ and by ctuity

$$\|Y(t)\| = \lim_{r\to\infty} \|Y_r(t)\| \leqslant \frac{1}{A} \|Y(0)\|$$

Take Y st $\|Y(0)\| = 1$ then $\|Y(t)\| \leqslant \frac{1}{A}$

Hence $\varphi(s) = \sup\{\|Y(0)\| : Y \in \mathcal{Y}_c \text{ st } \|Y(0)\| = 1\} \leqslant \frac{1}{A}$.

q.e.d.

Lemma 8:6. $\varphi(s+t) \leqslant \varphi(s).\varphi(t)$

Proof: For $Y \in \mathcal{Y}_c$, then $Z(t) = \dfrac{Y(t)}{\|Y(0)\|} \in \mathcal{Y}_c$ st $\|Z(0)\| = 1$.

Hence $\|Z(t)\| \leqslant \varphi(t)$ So $\|Y(t)\| \leqslant \|Y(0)\| \varphi(t)$

So given $Y \in \mathcal{Y}_c$ st $\|Y(0)\| = 1$ then:

$\|Y(t+s)\| \leqslant \|Y(s)\| \varphi(t) \leqslant \|Y(0)\| \varphi(s).\varphi(t)$.

Hence $\varphi(t+s) \leqslant \varphi(t).\varphi(s)$

q.e.d.

Lemma 8:7. If M admits no non-trivial normal $\underline{k}$-fields bdd above for all t then $\varphi(s) \to 0$ as $s \to \infty$.

Proof: (Modification of Eberlein)

φ is a positive function so we have to show that given any $\varepsilon > 0$ then there exists $T > 0$ st $\varphi(s) < \varepsilon$ for all $s > T$. Suppose not then there is $\varepsilon > 0$ and a sequence $\{t_n\} \to \infty$ st $\varphi(t_n) > \varepsilon$. Now $\varphi(s) = \sup\{\|Y(s)\| : \}$ so there is a sequence of normal $\underline{k}$-fields $\{Y_n\}$ along $\underline{k}$-lines $\{c_n\}$ st $\|Y_n(0)\| = 1$ and $\|Y_n(t_n)\| > \varepsilon$. $Y_n \in \mathcal{Y}_{c_n}$ and so by Cor6:7:1 there is L st $\|Y_n'(0)\| \leqslant L$ $\|Y_n(0)\| = L$ for L independent of the $\underline{k}$-line.

By compact property let $Y_n \to Y$ in subsequence. Lemma8:5:1 says $\varphi(s) < B$ for $s > 0$. Hence $\|Y_n(t+t_n)\| \leqslant B$ for $0 \leqslant t+t_n < \infty$. ie $-t_n \leqslant t < \infty$. By continuity and the fact that $t_n \to \infty$ then $\|Y(t)\| \leqslant B$ for all t.

So Y(t) is a normal $\underline{k}$-field on a $\underline{k}$-line bdd above for all t contradicting the hypothesis.

q.e.d.

For estimation we use:

Lemma (Eberlein p34) Let $\varphi : \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ satisfy:
(i) $\varphi(s) < B$. (ii) $\varphi(t+s) \leqslant \varphi(t).\varphi(s)$ (iii) $\varphi(s) \to 0$ as $s \longrightarrow \infty$
then there are numbers $a, b > 0$ st $\varphi(s) \leqslant ae^{-cs}$ for all $s \geqslant 0$.

<u>Proposition 8:8.</u> If M is a compact manifold, disconjugate for a <u>k</u>-flow admitting no non-trivial normal <u>k</u>-fields bdd. above for all t then the flow is Anosov.

Proof: By Lemmas 8:5, 8:6, 8:7 and the estimate above then

$$\varphi(s) \leqslant ae^{-cs} \quad \text{for all } s \geqslant 0.$$

Take $Y \in \mathcal{Y}_c$ then $\|Y(t)\| \leqslant \|Y(0)\| \varphi(t) \leqslant a \|Y(0)\| e^{-ct}$
Now consider the reverse flow and $Y^{-} \in \mathcal{Y}_{-c}$. The same hypotheses hold, since the reverse of a <u>k</u>-field is a <u>k</u>-field.

Hence $\|Y^{-}(t)\| \leqslant a_1 \|Y^{-}(0)\| e^{-c_1 t}$ for all $t \geqslant 0$ and all $Y^{-} \in \mathcal{Y}_{-c}$.

By prop^n 6:12 the flow is Anosov.

q.e.d.

Hence we have:

<u>Theorem 8:9.</u> Given a compact manifold M disconjugate for a <u>k</u>-flow then the flow is Anosov iff M admits no non-trivial normal <u>k</u>-fields, Y, st $\|Y(t)\|$ is bdd above for all t.

To complete this section we noted that if Y is a normal $\underline{k}$-field st $Y(0) = 0$, then $\|Y(t)\| \geq A \|Y(s)\|$ for $t \geq s \geq 1$ is not sufficient to guarantee Anosovity, the geodesic case. However proofs analogous to Eberlein([7]) enable us to show:

A $\underline{k}$-flow is Anosov if:

(i) there is $A > 0$ st for all non-trivial normal $\underline{k}$-fields on M with $Y(0) = 0$ and $\|Y(t)\| \geq A \|Y(s)\|$ for all $t \geq s \geq 1$.

(ii) $\int_1^{\infty} \frac{dt}{g(t)} < \infty$.

Section 9: Some Remarks on families of $\underline{k}$-flows on manifolds of negative curvature.

Up to this point we have only considered a single $\underline{k}$-flow on a manifold, with a given set of normals $N_{\underline{k}}$. We should like to ask if we can keep the N_i and alter the $\underline{k}$, analogous to looking at a magnetic field, keeping the normals but reducing the magnetic field strength. In particular if a manifold M is of sufficient curvature (negative) to ensure that a given $\underline{k}$-flow is Anosov, by decreasing the k_i do we remain Anosov, since the limiting case, $\underline{k} = 0$, the geodesic flow, is Anosov?

To compare two flows we use the following idea:

Definition: Given two flows φ_t, ψ_t on a manifold, N, then the flows are (topologically) conjugate if there is a homeomorphism $h: N \to N$ st
$$\begin{tikzcd} N \arrow[r, "\varphi_t"] \arrow[d, "h"'] & N \arrow[d, "h"] \\ N \arrow[r, "\psi_t"'] & N \end{tikzcd}$$
commutes.

ie sends orbits to orbits.

We can also ask if we perturb the flow do we alter its properties? Consider the set of all C^∞ flows on T^1M, say $\mathrm{Flow}(T^1M)$.

Definition: A flow is structurally stable if there is an open nhd V of the flow in $\mathrm{Flow}(T^1M)$ st any flow in V is conjugate to the original flow via a homeomorphism C^0 close to the identity.

Theorem (Anosov) An Anosov flow is structurally stable.

Proof: Anosov ([3]) p12.

Consider the C^1 topology: This says that two flows X,Y on T^1M are within ε of each other if given any $v \in T^1M$, letting c_1,c_2 be the integral curves of X,Y resp through v then

$$\left.\begin{aligned}\|\dot c_1(0)-\dot c_2(0)\| &< \varepsilon\\ \|\ddot c_1(0)-\ddot c_2(0)\| &< \varepsilon\end{aligned}\right\} \text{ for all } v \in T^1M.$$

Consider now two $\underline{k}$-flows on T^1M;

$X ; T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus k_1N(x,v))$ with $\underline{k}$-section $N_{\underline{k}}$

$Y : T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus k_1'M(x,v))$ with $\underline{k}'$-section $M_{\underline{k}'}$

So the integral curves of X satisfy:

$$\nabla_v v = \dot c + \Gamma(v,v) = k_1N_1$$
$$\nabla_v^2 v = \ddot c + 3\,\Gamma(v,\dot v) + \Gamma(\Gamma(v,v),v) + D\Gamma(v,v,v).$$

We can thus restate the C^1 norm in terms of the covariant derivative, by making

$$\|\nabla_{\dot c_1}\dot c_1 - \nabla_{\dot c_2}\dot c_2\| \leqslant K$$
$$\|\nabla^2_{\dot c_1}\dot c_1 - \nabla^2_{\dot c_2}\dot c_2\| \leqslant L$$

and asking that K,L be as small as possible.

<u>Proposition 9:1</u>. On a compact manifold of strict negative curvature there is a number $K \geqslant 0$ st if there is a $\underline{k}=(k_1,\ldots,k_n)$ flow on M with $k_1^2 + k_2^2 \leqslant K$ then it is conjugate to the geodesic flow on M.

Proof: Each line of the geodesic flow satisfies $\nabla_v v=0$, $\nabla_v^2 v = 0$ while the $\underline{k}$-flow satisfies $\nabla_v v = k_1N_1$ and $\nabla_v^2 v = k_1k_2N_2 - k_1^2 v$. Hence

$$\|\nabla_{\dot c_1}\dot c_1 - \nabla_{\dot c_2}\dot c_2\| = k_1$$
$$\|\nabla^2_{\dot c_1}\dot c_1 - \nabla^2_{\dot c_2}\dot c_2\| = k_1k_2 - k_2^2.$$

The geodesic flow is Anosov and hence stable, so there is an $\varepsilon > 0$ st any flow within ε of the geodesic is conjugate to it. Hence we can adjust the values of k_1 and k_2 say by looking at $k_1^2+k_2^2$, and we can make the distance from the geodesic flow as close as we need.

q.e.d.

We attempt to answer two questions:

(i) Given a self adjoint k-flow on a manifold with k-section N say can we reduce the k_i and still keeping the same N still have a k'- flow?

(ii) Given this type of k-flow as we reduce the k_i to zero, the geodesic case, are all the flows conjugate? Alternatively as we increase from the geodesic case how far can we go and still retain the Anosov properties?

To this end we examine the equations for normal k-fields and try to establish which parts will guarantee these properties. Unfortunately we have to modify the definition of Disconjugacy number and Anosov number, to enable comparisons as k alters. Hence for considering a single k-flow we use the definitions of section7, while for comparing two flows weuse the adapted definitions. In fact the new definitions are not novel but special cases of the old.

Suppose we return to the conditions on an arbitary section $\gamma : T^1M \longrightarrow F^{r+1}M$ thatmake it a k-section, section2. In particular consider the k-flow on the Lobachevsky plane. By the computation in Lemma2:3 $N : T^1M \longrightarrow F^2M$

generated the <u>k</u>-flow iff

$DN(x,v)(v,\ kN - \Gamma(x)(v,v)) + \Gamma(N,v) = -kv.$

Subsequent computation shows that N satisfies:

(i) $D_xN(v) - D_vN(\Gamma(v,v)) + \Gamma(v,N) = 0.$

(ii) $kD_vN(N(v)) = -kv.$

ie we have separated the parts out that depend on k. In the above case if we altered k the above would still hold and we still have a <u>k</u>-section.

To produce analogous results we need:

$p : F^{r+1}M \longrightarrow T^1M : (x,v,\zeta_1,\dots,\zeta_r) \longmapsto (x,v)$

$i : F^{r+1}M \longrightarrow TF^{r+1}M : (x,v,\zeta_1,\dots,\zeta_r) \longmapsto (x,v,\dots,\zeta_r)(v \oplus 0 \oplus \dots \oplus 0)$

Definition: A section $\zeta : T^1M \longrightarrow F^{r+1}M$ is compatible if $T(\zeta o p) o (i o \zeta) = (i o \zeta)$. ie

$$\begin{tikzcd}
F^{r+1}M \arrow[r, "i"] & TF^{r+1}M \arrow[r, "Tp"] \arrow[d, "\pi_1"] & TT^1M \arrow[d, "T\zeta"] \\
T^1M \arrow[u, "\zeta"] \arrow[r, "\zeta"] & F^{r+1}M \arrow[r, "i"] & TF^{r+1}M
\end{tikzcd}$$

commutes.

<u>Lemma 9:2</u>. A section $N : T^1M \longrightarrow F^{r+1}M : (x,v) \longmapsto (x,v,N_1,\dots,N_r)$ is compatible iff $DN_i(x,v)(v, -\Gamma(x)(\cdot v,v)) = -\Gamma(x)(v, \Gamma(x)(N,v))\ \forall i.$

Proof: $TNoTpoi(x,v,N_1,\dots,N_r) =$

$(x,\dots,N_r)(v \oplus 0 \oplus \dots \oplus DN_i(v) + \Gamma(v,N_i) \oplus \dots\dots)$

$i(x,v,\dots,N_r) = (x,\dots,N_r)(v \oplus 0 \oplus \dots \oplus 0).$

See appendix A for TN in covariant co-ordinates.

q.e.d.

Definition: For each $\underline{k}$ let $j_{\underline{k}} : F^{r+1}M \longrightarrow TF^{r+1}M$ by

$$j_{\underline{k}} : (x,v,\mathfrak{J}_1,\ldots,\mathfrak{J}_r) \longmapsto (x,\ldots,{}_r)(0 \oplus k_1\mathfrak{J}_1 \oplus \ldots \oplus k_{i+1}\mathfrak{J}_{i+1} - k_i\mathfrak{J}_{i-1} \oplus \ldots.)$$

<u>Lemma 9:3.</u> A compatible section, N, is the $\underline{k}$-section of a $\underline{k}$-flow iff

$$\begin{tikzcd} F^{r+1}M \arrow[r, "j_k"] & TF^{r+1}M \arrow[r, "Tp"] \arrow[d, "\pi_1"] & TT^1M \arrow[d, "TN"] \\ T^1M \arrow[u, "N"] \arrow[r, "N"] \arrow[ur, phantom, "\circlearrowleft"] & F^{r+1}M \arrow[r, "j_k"] \arrow[ur, phantom, "\circlearrowright"] & TF^{r+1}M \end{tikzcd}$$

Proof: Recall from section 2 the maps:

(i) $X_{\underline{k}} : F^{r+1}M \longrightarrow TF^{r+1}M$ generating the $\underline{k}$-flow on $F^{r+1}M$.

(ii) $Z_{\underline{k}} : T^1M \longrightarrow TT^1M : (x,v) \longmapsto (x,v,v \oplus k_1N_1(x,v))$

then $X_{\underline{k}} = i \oplus j_{\underline{k}}$ adding on fibres.

$$Tp\circ(i+j_{\underline{k}})\circ N(x,v) = Tp(v \oplus k_1N_1 \oplus \ldots \oplus -k_rN_{r-1})$$
$$= (v \oplus k_1N_1) = Z_{\underline{k}}(x,v)$$

By Prop^n 2:5 N is a $\underline{k}$-section iff $TN\circ Z_{\underline{k}} = X_{\underline{k}} \circ N$.

iff $TN\circ Tp\circ(i+j_{\underline{k}})\circ N(x,v) = (i+j_{\underline{k}})\circ N(x,v)$

$TN\circ Tp$ is linear on fibres.

iff $(TN\circ Tp\circ i\circ N)(x,v) + (TN\circ Tp\circ j_{\underline{k}}\circ N)(x,v) =$

$(i\circ N)(x,v) + (j_{\underline{k}}\circ N)(x,v)$ adding in fibres.

iff $TN \circ Tp \circ j_{\underline{k}} \circ N = j_{\underline{k}}\circ N$ since N is compatible.

q.e.d.

<u>Definition</u>: If $\underline{k} = (k_1,\ldots,k_r)$, $\underline{k}' = (k_1',\ldots,k_r')$ are positive then $\underline{k} \sim \underline{k}'$ if $k_i/k_1 = k_i'/k_1'$ for all i.

We can now extend our analogy of reducing the magnetic field strength.

<u>Lemma 9:4.</u> If a compatible section is the $\underline{k}$-section of a $\underline{k}$-flow then it isthe $\underline{k}'$section of any $\underline{k}'$-flow st $\underline{k} \sim \underline{k}'$.

Proof: If N is a $\underline{k}$-section then by the work in lemma2:3 $TN_{\underline{k}} \circ Z_{\underline{k}} = X_{\underline{k}} \circ N$ implies that in the fibres of $TF^{r+1}M$

$$(v \oplus e_2 + \Gamma(v,v) \oplus DN_1(x,v)(v,e_2) + \Gamma(N_1,v) \oplus \ldots\ldots \oplus DN_r(v,e_2) + \Gamma(N_r,v))$$

$$=$$

$$(v \oplus k_1N_1(x,v) \oplus \ldots \oplus k_{i+1}N_{i+1} - k_iN_{i-1} \oplus \ldots \oplus -k_rN_{r-1})$$

where $e_2 = k_1N_1(x,v) - \Gamma(x)(v,v)$.

ie. $$D_xN_i(v) + k_1D_vN_i(N_1) - D_vN_i(\Gamma(v,v)) + \Gamma(N_i,v) = k_{i+1}N_{i+1} - k_iN_{i-1}.$$

N is compatible, therefore by lemma9:2 and linearity

$$k_1D_vN_i(N_1) = k_{i+1}N_{i+1} - k_iN_{i-1}$$

Let $\underline{k}' \sim \underline{k}$ then dividing by k_1'/k_1

$$k_1'D_vN_i(N_1) = k_{i+1}'N_{I+1} - k_i'N_{i-1}$$

Substituting back into the equations, noting that the compatible part is independent of $\underline{k}, \underline{k}'$, we obtain the same equations with the k_i replaced by k_i'.

ie. $TN_{\underline{k}'} \circ Z_{\underline{k}'} = X_{\underline{k}'} \circ N$ so N is a $\underline{k}'$- section.

q.e.d.

<u>Corollary 9:4:1</u> If $N: T^1M \to F^2M$ is compatible and the $\underline{k}$-section of a $\underline{k}=(k,0,..,0)$ then it is a $\underline{k}'$-section for all $\underline{k}' = (t,0,..,0)$.

Proof. In the above N is a $\underline{k}$-section iff

$$k_1 D_v N(N) = -k_1 v \quad \text{and hence for all } k,$$

if for one k.

<u>Examples</u>. (a) On the Lobachevsky plane taking right or left hand normals. See examples in section 2.

(b) On a 2n manifold if $N : T^1M \to F^2M$ by

$$N(x,v) = \left(x,\ v,\ \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} v \end{bmatrix}\right).$$

$$\text{then } DN(x,v).N(x,v) = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}^2 \begin{bmatrix} v \end{bmatrix} = -v.$$

So gives a $\underline{k} = (k,0,..,0)$ flow for all k.
More generally if $N(x,v) = (x,v,Av)$ for constant matrix A st $A^2 = -I$.

<u>Corollary9:4:2</u>. Any section on an orientable surface of constant negative curvature is compatible and a $\underline{k} = (k)$ section for all k.

Proof: Compatible by example (a) above and a $\underline{k}$-section since it is of the form:

$$(x_1,x_2,v_1,v_2) = \left(x_1,x_2,v_1,v_2,\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \text{ or } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

since the section is smooth.

From this point on we shall assume that if two $\underline{k}$-flows are compared then they have common $\underline{k}$-section.

To reduce the 'field strength' we need to make the estimates of section 5. For example given a $\underline{k}$-flow on an oriented surface of constant negative curvature $-K^2$, then the Anosov number is $2k^2-K^2 < 0$ if $K^2 < 2k^2$. So if we reduce k to zero then the Anosov number remains negative, and hence we obtain:

Corollary 9:4:3. Given a surface of constant negative curvature $-K^2$ then all $\underline{k}$-flows are Anosov for $k^2 \leqslant \frac{1}{2}K^2$.

To extend this result we have to use the estimates for E,D,R as before. We can split E into:

$$E = E_1 + E_2 \quad \text{where } E_1 = \langle KoTN(F_j \oplus 0), F_i \rangle$$
$$E_2 = \langle KoTN(0 \oplus \nabla_v F_j), F_i \rangle$$

D and E_1 only depend on the section N, while E_2 contains terms in the k_i.

Recall that we were interested in the matrices:

$$M + k_1 N = \begin{bmatrix} k_1^2 & \\ & \Omega \end{bmatrix} + k_1(\tfrac{1}{2}(BD + D^T B^T) + \tfrac{1}{4}D^T D - \tfrac{1}{2}(E+E^T))$$

Definition: Let $k_1^2 [\,\Omega\,] + k_1(\frac{1}{2}(BD+D^T B^T) - \frac{1}{2}(E_2+E_2^T)) = G$

$$\tfrac{1}{4}(D^T D) - \tfrac{1}{2}(E_1+E_1^T) = H$$

So $M+k_1N = G + k_1H$ where G depends on $\underline{k}$, H does not.

We introduce a slightly altered disconjugacy number based on this new splitting.

Definition: Given a self adjoint $\underline{k}$-flow on a manifold, M, then if there are numbers K_1^2, K_2^2, st along any $\underline{k}$-line and for any completion $G(v) \leqslant K_1^2 I$, $H(v) \leqslant K_2^2 I$ then

$$K_4^2 = K_1^2 + k_1 K_2^2 \quad \text{is the disconjugacy number.}$$

Proposition 9:5. On compact M $K_4^2 < \infty$ and $M + k_1 N \leqslant K_4^2 I$.

Proof: By construction $M + k_1 N = G + k_1 H \leqslant (K_1^2 + k_1 K_2^2)\, I = K_4^2 I$

Apply the same estimates as in Corollary5:10:1

$\frac{1}{2}(E_1+E_1^T) < E(v) < E < \infty$

$\frac{1}{2}(E_2+E_2^T) < E(v) < E < \infty$

So by same analysis $K_4^2 < \infty$

q.e.d.

If K_0^2 is the disconjugacy number as defined in section 5 then the above gives:

Corollary9:5:1. $K_0^2 \leqslant K_4^2 < \infty$.

Proposition 9:6. If a compact manifold of negative curvature has curvature in $[-K_2^2, -K_1^2]$ then a $\underline{k}$-flow on M is disconjugate if $K_1^2 \geqslant K_4^2$.

Proof. By above $K_1^2 \geqslant K_4^2 \geqslant K_0^2$ and hence by Prop5:11 the manifold is disconjugate for the flow.

q.e.d.

Definition: If $\underline{k} = (k_1, \ldots, k_r)$, $\underline{k}' = (k_1', \ldots, k_r')$ put $\underline{k} \leqslant \underline{k}'$ iff $\underline{k} \sim \underline{k}'$ and $k_1 \leqslant k_1'$.

Proposition 9:7. Given a $\underline{k}$-flow on a compact manifold M with disconjugacy number K_4^2 and a compatible $\underline{k}$-section then the $\underline{k}'$- flow has disconjugacy number $(K_4')^2 \leqslant K_4^2$ for all $\underline{k} \geqslant \underline{k}'$.

Proof: If the flow has disconjugacy number K_4^2 then there are numbers st $G \leqslant K_1^2 I$ $H \leqslant K_2^2 I$. Given another $\underline{k}' \leqslant \underline{k}$ then $H = H'$ by construction, and $k_1' H \leqslant k_1 H \leqslant k_1 K_2^2 I$.

$$\text{Consider } G = \begin{bmatrix} k_1^2 & \\ & \underline{0} \end{bmatrix} + k_1\left(\tfrac{1}{2}(BD+D^TB^T) - \tfrac{1}{2}(E_2'+E_2'^T)\right)$$

Since the $\underline{k},\underline{k}'$ sections are the same $D=D'$.

Hence

$$G' = \begin{bmatrix} k_1'^2 \; --- \; 0 \\ \underline{0} \end{bmatrix} + k_1'\left(\tfrac{1}{2}(B'D+D^TB'^T) - \tfrac{1}{2}(E_2' + E_2')\right)$$

$$B' = \begin{bmatrix} 0 & -k_2' & 0 & & & & \\ k_2' & 0 & -k_3' & & & & \\ & & \ddots & & & & \underline{0} \\ & & & 0 & -k_r' & & \\ & & & k_r' & 0 & & \\ & & \underline{0} & & & & \underline{0} \end{bmatrix} = aB \qquad \text{for some } a$$

$$(E_2')_{ij} = k_{j+1}' \langle D_v N(F_{j+1}), F_i \rangle - k_j' \langle D_v N(F_{j-1}), F_i \rangle$$

$$= a(E_2)_{ij}.$$

So $G' = a^2 G$ for $0 < a \leqslant 1$.

So $G' + k_1 H' \leqslant a^2(K_1^2 + k_1 K_2^2) I \leqslant aK_4^2 I \leqslant K_4^2 I$

So for any completion $(K_4')^2 \leqslant K_4^2$.

q.e.d.

We similarly define a revised Anosov number.

By Prop 5;9 we have the following:

$D = \sup \{D(x,v,F(v))$; over all $\underline{k}$-lines, all completions$\}$

$E_i = \sup \{E_i(x,v,F(v))$; " " " " " " $\}$ $i=1,2$

On a compact manifold we defined

$$K_3^2 = (n-1)^2 D L_1^2 + 2k_1^2 + k_1 E$$ [see the estimates in section 7.]

Definition: For a self adjoint $\underline{k}$-flow with compatible $\underline{k}$-section then let:

$$K_3^2 = (n-1)^2 L_1^2 + 2k_1^2 + k_1(E_1+E_2) \quad \text{where}$$
$$L_1^2 = (n-1)^{5/2} K_1 + |B| + \tfrac{1}{2} k_1 D$$

We can then define:

Definition: Given a $\underline{k}$-flow on a compact manifold of negative curvature, with curvatures in $[-K_2^2, -K_1^2]$, and a compatible $\underline{k}$-section then let $K = \max\{K_4^2 - K_1^2, K_3^2 - K_1^2\}$ be the Anosov number.

Lemma 9:8. A $\underline{k}$-flow on a compact manifold M of negative curvature, with compatible $\underline{k}$-section is Anosov if $K < 0$.

Proof:: If M has curvature in $[-K_2^2, -K_1^2]$ with disconjugacy number then the flow is disconjugate as before.

We also require the constant L_0 in prop^{n}7:2 to be positive. Since $E \leqslant E_1 + E_2$ then $K_3^2 - K_1^2 < 0$ implies that $(n-1)^2 D L_1^2 + 2k_1^2 + k_1 E - K_1^2 > 0$ and hence by the analysis in propns 7:3, 7:2 the flow is Anosov.

q.e.d.

If K' is the Anosov number defined as in section 7 then the above shows:

Corollary 9:8:1 $K \leqslant K'$.

Lemma 9:9. Given a compact manifold M of negative curvature admitting a compatible, self adjoint $\underline{k}$-flow with Anosov number K then if $\underline{k}' \leqslant \underline{k}$ the Anosov number, K', of the $\underline{k}'$- flow satisfies $K' \leqslant K$.

Proof: By prop^n 9:7 $(K_4')^2 \leqslant K_4^2$ disconjugacy number

Hence $(K_4')^2 - K_1^2 \leqslant K_4^2 - K_1^2$.

By the same work in prop^{n}9:7 we have since $\underline{k}' \leqslant \underline{k}$ $B' = aB$, $E_2' = aE_2$ for $0 < a \leqslant 1$, and $D' = D$, $E_1' = E_1$.

So

$$\begin{aligned}(K_3')^2 &= (n-1)^{5/2} D L_1'^2 + 2k_1'^2 + k_1'(E_1' + E_2') \\ &\leqslant (n-1) D L_1^2 + 2k_1^2 + k_1(E_1 + E_2) = K_3^2\end{aligned}$$

So $(K_3')^2 - K_1^2 \leqslant K_3^2 - K_1^2$

So $$\begin{aligned}K' &= \max((K_3')^2 - K_1^2), (K_4')^2 - K_1^2) \\ &\leqslant \max(K_3^2 - K_1^2, K_4^2 - K_1^2). = K.\end{aligned}$$

q.e.d.

Theorem 9: 10. Given a compact manifold M of negative curvature admitting a compatible $\underline{k}$-flow with Anosov number $K < 0$, then any $\underline{k}'$- flow on M with $\underline{k}' \leqslant \underline{k}$ is Anosov.

Proof: By lemma 9:8 and lemma 9:9.

q.e.d.

Hence if we can find a compatible self adjoint $\underline{k}$-flow with Anosov number < 0 then we have a whole family of $\underline{k}$-flows down to the geodesic flow, all Anosov. We can now ask if the relationship is stronger than this.

Proposition 9:11. A self adjoint $\underline{k}$-flow with compatible $\underline{k}$-section on a manifold of negative curvature with Anosov number < 0 is conjugate to the geodesic flow.

Proof: If $\underline{k} = (k_1, \ldots, k_r)$ with section N consider $t\underline{k} = (tk_1, \ldots, tk_r)$. Since the section is compatible we can consider the $t\underline{k}$-flow on M, for $0 \leqslant t \leqslant 1$, $X_{t_{\underline{k}}}$ say.

Hence consider in the space of flows on T^1M the path

$$c : [0,1] \longrightarrow \mathrm{Flow}^r(T^1M) : t \longmapsto X_{t_{\underline{k}}}$$

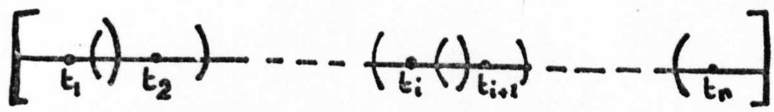
Hence c(0) is the geodesic flow, c(1) the $\underline{k}$-flow and c(t) a $\underline{k}'$-flow for $\underline{k}' \leqslant \underline{k}$. Hence by lemma9:8 all flows on the path are Anosov. Moreover c is cts, since they all have common $\underline{k}$-section and hence if $t \neq s$ let v_s be $v \in T^1M$ under the $s\underline{k}$ flow etc then

$$\left\| \nabla_{v_s} v_s - \nabla_{v_t} v_t \right\| = \left\| (tk_1 - sk_1)N_1 \right\| = (t-s)k_1$$

$$\left\| \nabla^2_{v_s} v_s - \nabla^2_{v_t} v_t \right\| \leqslant (t^2-s^2)(k_1k_2+k_1^2)$$

Hence given $\varepsilon > 0$ we can arrange s,t st in the C^1 topology the $t\underline{k}$ and $s\underline{k}$ flows are within ε.

By the theorem of Anosov at the beginning of the section all the flows on the curve are structurally stable since they are all Anosov. Hence for all $0 \leqslant t \leqslant 1$ there is $\varepsilon(t)$ st all flows on c with $(t-s) < \varepsilon(t)$ are conjugate. Hence we have a covering of $[0,1]$ by $\varepsilon(t)$ nhds. $[0,1]$ is compact, so choose a finite subcover and hence a sequence of numbers $0 \leqslant t_1 < \dots < t_i < \dots < t_n \leqslant 1$ st



Moreover $c(t_i)$ is conjugate to $c(t_{i+1})$, since we can choose $t \in U_i \cap U_{i+1}$ and by stability $c(t_i) \cong c(t) \cong c(t_{i+1})$ by a homeomorphism h_i say

$$\begin{tikzcd} M \arrow[r] \arrow[d, "c(t_i)"'] & M \arrow[r] \arrow[d, "c(t)"] & M \arrow[d, "c(t_{i+1})"] \\ M \arrow[r] & M \arrow[r] & M \end{tikzcd}$$

Then $h = h_{n+1} o h_n o \dots o h_1 o h_0$ is the conjugating homeomorphism twixt thegeodesic and the $\underline{k}$-flow, since aech step of the ladder below commutes:-

$$\begin{tikzcd} M \arrow[r, "h_0"] & M \arrow[r, "h_1"] & M \arrow[r] & \cdots \arrow[r] & M \arrow[r, "h_n"] & M \\ M \arrow[r, "h_0"'] \arrow[u, "c(0)"] & M \arrow[r, "h_1"'] \arrow[u, "c(t_1)"] & M \arrow[r] \arrow[u, "c(t_2)"] & \cdots \arrow[r] & M \arrow[r, "h_n"'] \arrow[u, "c(t_n)"] & M \arrow[u, "c(1)"] \end{tikzcd}$$

q.e.d.

Note: Each of the individual h_i is C^0 close to the identity but the compounded h need not be.

We can sum up the last two theorems on altering $\underline{k}$ thus:-

Theorem 9:12. Given a self adjoint $\underline{k}$-flow on a manifold of negative curvature with compatible $\underline{k}$-section and Anosov number $K<0$, then given $\underline{k}' \leqslant \underline{k}$ the $\underline{k}'$- flow with same section is Anosov and conjugate to the $\underline{k}$- flow.

Proof: Follows, since by above both flows are conjugate to the geodesic flow.

<u>Corollary 9:12:1</u>: Given a compact oriented surface of constant negative curvature $-2K^2$, then any <u>k</u>- flow on the surface with $k < K^2$. is Anosov and conjugate to any other <u>k</u>$'$-flow with $k' \leq k$.

Proof: Corollaries 9 : 4 :(1,2,3) and th^m 9:12.

It would be interesting to find a <u>k</u>-flow on a manifold of negative curvature with Anosov number < 0 but no compatible <u>k</u>-section or conjugate to the geodesic flow. On could conjecturethat these are 'isolated' cases in the sense of the paths of flows in $\mathrm{Flow}^r(T^1M)$ and that a compatible <u>k</u>- section is a more usual case.

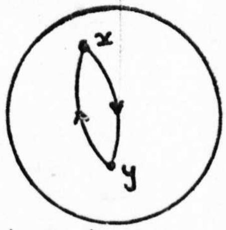
Section 10: Some Conjectures and Ideas for future work.

Many results on the geodesic flow have been on manifolds of negative curvature, for instance on a simply connected manifold of negative curvature there is a unique geodesic between any two distinct points.(Hicks [13]). This is proved by looking at the Jacobi fields and the disconjugacy condition. Unfortunately it also relies heavily on the 'straight' line nature of a geodesic, not generalisable to our case.We have however for compatible k-sections and small enough k, a homeomorphism, h, between the k-flow and the geodesic flow, sending k-lines to geodesics, which may be of use in giving more information on the geometry of k-lines.The stumbling block at present is the fact that h is a homeomorphism, not an isomorphism preserving the fibres of T^1M. If $h(x,v) = (g(x),h(x,v))$ then given two points x,y in M we could obtain a k-line of the flow by taking the image of the geodesic through g(x) and g(y).

Even if we could find a k-line between x and y, giving a geodesic between g(x),g(y), suppose we reverse this geodesic and consider the k-line giving this geodesic; it may not, pointwise, be the original k-line.

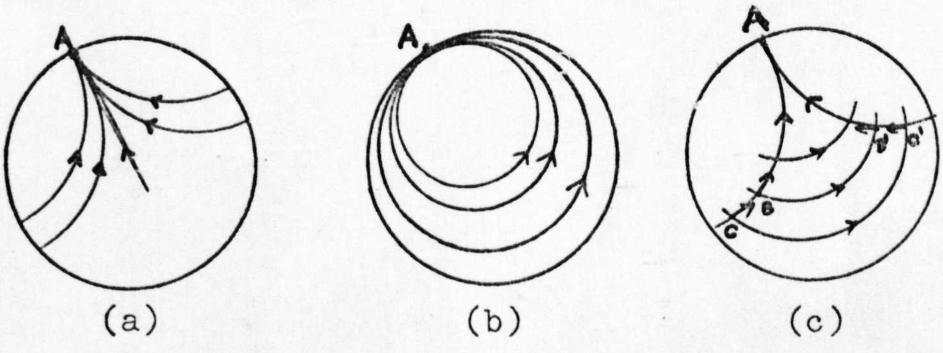
Conjecture: Given a reversible k-flow on a manifold of negative curvature with Anosov number $K < 0$, then there is a unique k-line of the flow between any two distinct points of the manifold, and the flow is isomorphic to the geodesic flow.

Another possibly interesting class of k-flows might be when the flow is not reversible, but given two points x,y, on M with a flow line say from x to y then there is a flow line, probably not the same, from y to x . ie the flow is 'returnable', such as the flow on the Lobachevsky plane:-



A flow as above, returnable but not reversible cannot have fibre preserving homeomorphisms, but we could conjecture that on a manifold of strict negative curvature, which is simply connected, then given two distinct points there are unique k-lines in the oriented directions between the points, as in the case above, for a range of k-flows.

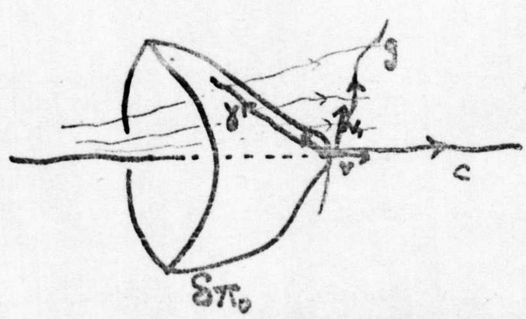
There is also another possibly fruitful analogue to properties of the geodesic flow that may be investigated. If we return to the geodesic flow on the Lobachevsky plane then there is the so called horocycle flow on the plane, which ergodically has opposite properties to the geodesic and flow lines as in (b).



(a) (b) (c)

The interesting property is that given a pencil of geodesics as in (a) then drawing in the horocycle family through the same 'absolute' point A we can set for each horocycle H_0 the set $\mathcal{H}_0 = \{(x,v) \in T^1M :$ $x \in H_0$ and v is the velocity of the geodesic cutting H_0 at $x\}$ Similarly for H_1. If φ_t is the geodesic flow then $\varphi_t(\mathcal{H}_0) = \mathcal{H}_1$.

It turns out that $\{\mathcal{H}_0\}$ is a foliation of T^1M, invariant and contracting under φ_t, with $\mathcal{H}_0$ tangent to E^s_v at v. The expanding geodesics give a similar expanding foliation. Anosov has shown that any Anosov flow admits such contracting and expanding foliations, in the case of the geodesic flow precisely the horocycles. We should like to reverse the above and by considering the contracting foliation for an Anosov k-flow produce 'horocycle' flows. Given the leaf $\mathcal{H}_0$ of the foliation we can locally project it to a small 'disc', $S\pi_0$, in M about the flowline, c. Consider the normals $N_1(v)$, along the k-lines giving this disc. N_1 at v may not be tangential to the disc, which is only transversal to the flowline, but we can 'project' N_1 onto a vector tangential to $S\pi_0$ using the flowlines. Let g be the geodesic with initial velocity N_1. Through each point of $S\pi_0$ we have a k-line formed from the projection of $\mathcal{H}_0$, which close to the disc must intersect the geodesic since the disc is n-1 dimensional. Hence the geodesic will project onto a curve γ on the disc by pulling back along the flow lines. The velocity of this curve at $\gamma(0)$ will give a vector tangential to $S\pi_0$. Piecing these local discs together should give a vectorfield on each leaf of the foliation, and hence curves in M which lift to lines in T^1M, one line for each $v \in T^1M$. In the case of the geodesic flow this yields the horocycle flow.



we did the above for N_1. Doing this for any section $\mathcal{J}$: $T^1M \quad F^2M$ should give a range of horocycle flows, for any given k-flow. This promotes the question that since the k-lines in the non-geodesic normal directions behave like geodesics; if we took such a section at all points perpendicular to the k-section would we generate the same horocycle flow as with the geodesic flow?

A further line of approach might be to look at k-flows from a projective geometry point of view. The Lobachevsky plane is the INSIDE of the unit circle. Adding the boundary circle, the Absolute, adds in a set of 'infinite' points. If one stood at A and looked along the straight lines, geodesics, then the horocycles, H_0, are the horizons of vision (horos horizon gk.).

Eberlein has generalised the Absolute concept for the geodesic flow on visibility manifolds, [simply connected manifolds of strict negative curvature], by adding in points at infinity; where the horocycles become circles with centre at infinity. It should be possible to generalise this to arbitary k-flows though the topology of the new space depends on the straight line properties of the geodesics, which as yet we do not have.

Finally the obvious next step in the study should be of the non self adjoint k-flows and the case of variable k, say with k depending on position on M, k = k (m). The variation equations can be set up in matrix form precisely as we did before; however it is the solution of the matrix equations that is the drawback at the moment, but it should be possible at some future date to solve the equations as before. This will give a wide range of tangent flows.

APPENDICES:

A ; On the differential geometry of the unit tangent and frame bundles.

B ; Some concepts from Classical Mechanics.

APPENDIX A.

The Differential Geometry of the Frame Bundles.

In Section 1 of this appendix we quote some results of Sasaki, |23| and |24|, and Eliasson, |10|, to provide a description of the differential geometry of the unit tangent bundle, T^1M, that is required in proving both the existence of, and the Anosovity of k-flows on a Riemannian manifold, M. In Section 2 we have extended these ideas and used them to study the differential geometry of the bundle of oriented orthonormal r-frames, F^rM over the manifold M, which is necessary for studying the existence of k-flows. In Section 3 we generalise the idea of the geodesic flow on the space of r-frames to one based on lines of known geodesic curvature. The only reference known for this Section is Arnold, |4|, where no proofs are provided, so the results of Sections 2 and 3 are an original description.

Section 1: The unit tangent bundle, $T'M$.

Suppose (U,φ) is a chart for M.

$\varphi: U \to U' \subset \mathbb{R}^n : u \mapsto (x^1(u), \cdots\cdots, x^n(u).)$

Then $(TU, T\varphi)$ is a chart for TM with a basis $\{\partial/\partial x^i\}_x$ for T_xM.

The Riemannian metric, g, is expressed by the quadratic

$$ds^2 = g_{ij}(x)dx^i dx^j$$

Consider also $TTM \to TM \to M$ and the maps:

$\pi_2 : TTU \to TU : (x, v, e, f) \mapsto (x, v)$

$T\pi_1 : TTU \to TU : (x, v, e, f) \mapsto (x, e)$

and the CONNECTOR MAP:

$K : TTU \to TU : (x, v, e, f) \mapsto (x, f + \Gamma^{(\varphi)}(x)(v,e))$

Then Eliasson, |10|, shows that the map

$\pi_2 \oplus T\pi_1 \oplus K ; TTU \to TU \oplus TU \oplus TU : (x,v, e, f) \mapsto ((x,v) \oplus (x,e) \oplus (x, f+\Gamma^{\varphi}(x)(v,e))$

is the local expression for a C^∞ diffeomorphism from TTM to $TM \oplus TM \oplus TM$. This means that in any fibre T_vTM the points may be represented by

$$T\pi_1 \oplus K : (x,v,e,f) \mapsto (x, v, e \oplus f + \Gamma^{\varphi}(x)(v, e))$$

or that TTU can be represented by COVARIANT CO-ORDINATES

$$\{ (x, v, \delta x \oplus \delta v) ; \delta x, \delta v \in T_xM \}$$

Henceforth quantities in TTM written with the $\oplus$ symbol will be in the covariant co-ordinates, those without in the usual co-ordinates.

In these new co-ordinates the connector map becomes a projection and we have the two maps:

$T\pi_1 : TTU \to TU : (x, v, \delta x \oplus \delta v) \mapsto (x, \delta x)$

$K ; TTU \to TU : (x, v, \delta x \oplus \delta v) \to (x, \delta v)$

Remark: We have found the following 'interpretation' of the splitting $T_vTM \cong T_xM \oplus T_xM$, under the maps $T\pi_1$ and K, useful in appreciating the role of TTM, and later TF^rM for the frame bundles.

Consider a curve $c : I \longrightarrow TM : (t) \longmapsto (x(t), v(t))$ and the velocity $\dot{c} : I \longrightarrow TTM : t \longmapsto (x, v, \dot{x}, \dot{v}) \equiv (x, v, \dot{x} \oplus \dot{v} + \Gamma(x)(v, \dot{x}))$

$T\pi_1$ picks out the velocity $\dot{x}$ of the carrier x, and hence the space spanned by the inverse image of T_xM under $T\pi_1$ is the <u>horizontal</u> subspace of T_vTM

K picks out the covariant velocity of the vector v, and the space spanned by the inverse image of T_xM under K (within fibres variation) is the <u>vertical</u> subspace of T_vTM.

Later we shall split each fibre $T_\xi F^rM$ similarly into a horizontal subspace, carrier velocity, and vertical subspaces, giving the 'velocity' of each frame element.

. . . .

We can now put a metric on TTM

Since δx and $\delta v \in T_xM$ then we have an inner product on T_vTM by

$$\langle\langle (\delta x \oplus \delta v), (\delta x' \oplus \delta v') \rangle\rangle_v = \langle \delta x, \delta x' \rangle_x + \langle \delta v, \delta v' \rangle_x$$

where $\langle\ ,\ \rangle_x$ is the inner product given by g in T_xM.

This induces a 'Pythagorean' norm on T_vTM by

$$|||(\delta x \oplus \delta v)|||_v^2 = \|(x, \delta x)\|_x^2 + \|(x, \delta v)\|_x^2$$

where $\|\ \|_x$ is the norm given by g in T_xM.

$|||\ |||$ is a norm given by a Riemannian metric G on TM with quadratic form:

$$d\sigma^2 = g_{ij}dx^i dx^j + g_{ij}\delta v^i \delta v^j.$$

where $\delta v^i = dv^i + \Gamma^i_{jk}(x)(v^j dx^k)$, a covariant term.

If in TU we write the co-ordinates as $(x^1 \ldots x^n, v^1 \ldots v^n)$ $= (x^1, \ldots\ldots x^{2n}) = \{x^I : I = 1, \ldots 2n\}$ then we can write the above as

$$d\sigma^2 = G_{IJ}\, dx^I dx^J$$

where G as a matrix looks like:

$$\left[\begin{array}{c:c} g^{ij} + g^{pq}\,\Gamma^{p}_{ri}\,{}^{q}_{sj}\,v^{r}v^{s} & [pi,j]v^{p} \\ \hdashline [pi,j]v^{p} & g^{ij} \end{array}\right] = G_{IJ} : I,J=1..2n$$

$$\left[\begin{array}{c:c} g^{ij} & -\Gamma^{i}_{rs}\,g^{js}v^{r} \\ \hdashline -\Gamma^{i}_{rs}\,g^{js}v^{r} & -g^{ij} + g^{pq}\,\Gamma^{j}_{pr}\,\Gamma^{i}_{qs}\,v^{r}v^{s} \end{array}\right] = G^{IJ}: \ I,J=1..2n$$

Later we shall want to look at T^1M and TT^1M, the unit tangent spaces. If $j: T^1M \to TM: (x,v) \mapsto (x,v)$ is the inclusion map then we can use $Tj: TT^1M \to TTM$ to induce a covariant chart system on TT^1M.

If we again have our chart (U, φ) for M then $(TT^1U\ TT^1\varphi)$ is the induced chart for TT^1M. If we again make use of the map $\pi_2 \oplus T\pi_1 \oplus K$ restricted to TT^1M we derive covariant co-ordinates for this manifold.

i.e. $\{(x, v, \delta x \oplus \delta v): \delta x, \delta v \in T_xM;\ v \in T^1_xM.\}$

So $\|v\|_x^2 = \langle v,v\rangle_x = 1$. Taking the covariant derivative this means that $0 = 2\langle v,\delta v\rangle$ or that:

TT^1M has COVARIANT CO-ORDINATE CHARTS $\{(TT^1U, TT^1\varphi)\}$ such that if $\xi \in T_vT^1M$ then

$$\xi = ((x, v, \delta x \oplus \delta v): v \in T^1_xM, \delta x, \delta v \in T_xM \text{ and } \langle v, \delta v\rangle = 0)$$

We also have the Riemannian metric on T^1M given by

$$\ll (\delta x \oplus \delta v), (\delta x' \oplus \delta v') \gg_v = <\delta x, \delta x'>_x + <\delta v, \delta v'>_x$$

$$|||(\delta x \oplus \delta v)|||_v^2 = \|(x,\delta x)\|_x^2 + \|(x,\delta v)\|_x^2$$

The effect of this splitting is to exchange ordinary differentiation for covariant differentiation; for instance suppose we have a vector field

$$X : T^1M \to TT^1M : (x,v) \mapsto (x,v,\dot{x},\dot{v}),$$

then in covariant co-ordinates this becomes

$$X : T^1M \to TT^1M : (x,v) \mapsto (x,v,\dot{x} \oplus \dot{v} + \Gamma(v,\dot{x})) = (x,v,\dot{x} \oplus \nabla_{\dot{x}} v)$$

In particular the geodesic flow takes the simple form

$$X : T^1M \to TT^1M : (x,v) \mapsto (x,v, v \oplus 0)$$

reflecting the fact that for gedesics $\nabla_v v = 0$.

. . . .

Section 2: The Frame Bundles F^rM, $r>1$.

We consider T^1M as F^1M and generalise the results of the previous section.

Given a chart (U,φ) for M consider

$$F_x^rU = \{ (x, v_1, \ldots, v_r) : v_i \in T_xU \text{ st. } <v_i, v_j> = \delta_j^i \}$$

and $F^rU = \{\bigcup F_x^r : x \in U\}$, $F^r\varphi = T\varphi \times T\varphi \times \ldots \times T\varphi$. r-times.

Then if we take the usual definition of the space of oriented r-frames F^rM as $\{ (v_1 \oplus v_2 \oplus \ldots \oplus v_r) \in TM \oplus \ldots \oplus TM : <v_i, v_j> = \delta_j^i \}$, F^rM forms a principal bundle over M with group $O(n)$ and with a manifold structure given by the charts $\{F^rU, F^r\varphi\}$.

We should like to develop a covariant co-ordinate system for TF^rM just as for TT^1M and so a Riemannian structure on F^rM.

Lemma A:1. Given the co-ordinate system $(F^rU, F^r\varphi)$ on F^rM then the co-ordinate system $(TF^rU, TF^r\varphi)$ can be represented as: $\{(x, v_1, \ldots, v_r)(x \oplus v_1 \oplus \ldots \oplus v_r)\}$ st $\delta v_i \in T_xU$ and $\langle v_i, \delta v_j\rangle + \langle v_j, \delta v_i\rangle = \delta^i_j$, for all i,j.

Proof.

Consider the inclusions:

$$F^rU \xrightarrow[i]{} TU \oplus TU \oplus \ldots \oplus TU \xrightarrow[j]{} TU \times TU \times \ldots \times TU \quad \text{then}$$

$$TF^rU \xrightarrow[Ti]{} T(TU \oplus TU \oplus \ldots \oplus TU) \xrightarrow[Tj]{} TTU \times TTU \times \ldots \times TTU$$

If we set $p_i : TM \times TM \times \ldots \times TM \to TM \to M$

$$:: (x,v_1), \ldots (x,v_r) \mapsto (x,v_i) \mapsto (x)$$

then $TM \oplus TM \oplus \ldots \oplus TM = \{(\xi_1, \ldots, \xi_r) \in TM \times \ldots \times TM : p_i(\xi_i) = p_k(\xi_k)\}$

i.e. j satisfies $p_i o j = p_k o j \ \forall i,k$.

Hence $T(TM \oplus \ldots \oplus TM) = \{(\eta_1, \ldots, \eta_r) \in TTM \times \ldots TTM : Tp_i(\eta_i) = Tp_k(\eta_k)\}$

i.e. $Tp_i o Tj = Tp_k o Tj \quad \forall i,k$.

Now $j : TU \oplus TU \oplus \ldots \oplus TU \to TU \times TU \times \ldots \times TU$

$$: ((x,v_1) \oplus \ldots \oplus (x,v_r)) \mapsto ((x,v_1), \ldots (x,v_r))$$

So $Tj : ((x,v_1) \oplus \ldots \oplus (x,v_r)(\eta, \xi_1, \ldots, \xi_r))$

$$\to ((x,v,p_1,q_1), \ldots (x,v,p_r,q_r)) \in TTU \times \ldots \times TTU$$

and the above implies that

(i) $(x,p_i) = (x,p_k)$ for all i,k.

(ii) $p_i = \eta$, $q_i = \xi_i$ for all i,k.

So $T(TU \oplus TU \ldots \oplus TU) = \{(x,v, \ , \ _1), \ldots (x,v, \ , \ _r)\}$

we can split each $TTU \cong TU \oplus TU \oplus TU$ as before. Hence $((x,v,\eta,\xi_1), \ldots (x,v,\eta,\xi_r))$ can be represented as

$$((x,v) \oplus (x,\eta) \oplus (x, \xi_1 + \Gamma^\varphi(v,\eta)), \ldots, ((x,v) \oplus (x,\eta) \oplus (x, \xi_r + \Gamma^\varphi(v,\eta))$$

or in a clearer notation

$$((x,v_1, \ldots v_r)(\eta \oplus \xi_1 + \Gamma^\varphi(v,\eta) \oplus \ldots \oplus \xi_r + \Gamma^\varphi(v,\eta)))$$

So $T(TU \oplus TU \ldots \oplus TU) = \{(x, v_1, \ldots v_r)(\delta x \oplus \delta v_1 \oplus \ldots \oplus \delta v_r)\}$

Now $i : F^r U \to TU \oplus TU \oplus \ldots \oplus TU$ implies that

$F^r U = \{(x, v_1, \ldots v_r) \in TU \oplus \ldots \oplus TU \text{ st. } \langle v_i, v_k \rangle = \delta^i_k\}$

Taking the covariant derivative as before we see that

$$0 = \delta\langle v_i, v_k\rangle = \langle v_i, \delta v_k\rangle + \langle \delta v_i, v_k\rangle$$

So we can represent $TF^r U$ as the set of points

$$\{(x, v_1, \ldots v_r)(\delta x \oplus \delta v_1 \oplus \ldots \oplus \delta v_r);\ \delta x, \delta v_i \in T_x U,\ \underset{=\delta^i_k}{\langle v_i, \delta v_k\rangle + \langle \delta v_i, v_k\rangle}\}$$

q.e.d.

Proposition A:2. $\{(TF^r U, TF^r \varphi)\}$ in the above notation is an atlas for $TF^r M$ for a given atlas $\{(U, \varphi)\}$ for M.

Proof We show that the representation above extends globally on the whole of $TF^r M$. If (U, φ) and (V, ψ) are charts for M such that $U \cap V \neq \phi$, then $TF^r U \cap TF^r V \neq \phi$. If we set $g = \psi\varphi^{-1}$ then the co-ordinate transformations for $TTU \to TTV$ induce a transformation $TF^r g : (TF^r\varphi)(F^r U) \to (TF^r\psi)(TF^r V)$ by

$$\begin{array}{ccc} TF^r\varphi\,(TF^r U \cap V) & \xrightarrow{T(ioj)} & TTU \times TTU \times \ldots \times TTU \\ \Big\downarrow{\scriptstyle TF^r g} & & \Big\downarrow{\scriptstyle TTg \times \ldots TTg} \\ TF^r\psi\,(TF^r U \cap V) & \xrightarrow{T(ioj)} & TTV \times TTV \times \ldots \times TTV \end{array}$$

Now $T(ioj)((x, v_1, \ldots v_r)(\delta x \oplus \delta v_1 \oplus \ldots \oplus \delta v_r)) =$
$((x, v_1, x, \delta v_1 - \Gamma^\varphi(x)(v_1, \delta x)), \ldots\ldots, (x, v_r, \delta x, \delta v_r - \Gamma^\varphi(x)(v_r, \delta x)))$.

and hence $(TTg \times TTg \times \ldots \times TTg)(T ioj)(x, \ldots v_i \ldots)(\delta x \oplus \ldots \oplus \delta v_i \ldots))$
$= (\ldots\ldots, (g(x), Dg(x)(v_i), Dg(x)(\delta x), A, \ldots\ldots\ldots)$ where
$A = (D^2 g(x)(v_i, v_i) + Dg(x)(\delta v_i - \Gamma^\varphi(x)(v_i, \delta x))$

Eliasson |10|, shows that the Christoffel symbols transform thus

$$\Gamma^{\psi}(g(x))(Dg(x)(v_i), Dg(x)(\delta x)) + D^2g(x)(v_i, v_i) = Dg(x)(\Gamma^{\varphi}(x)(v_i, \delta x)$$

,

So if we set $TF^r g((x, v_1, \ldots v_r)(\delta x \oplus \delta v_1 \ldots \oplus \delta v_r)) =$
$((g(x), \ldots, Dg(x)(v_i), \ldots)(Dg(x)(\delta x) \oplus \ldots \oplus Dg(x)(\delta v_i) \oplus \ldots))$
then $T(i\circ j)(TF^r g) = (TTg \times \ldots \times TTg)(T(i\circ j))$ and the above diagram commutes.

Moreover if $((x, v_1, \ldots, v_r)(\delta x \oplus \delta v_1 \oplus \ldots \oplus \delta v_r)) \in TF^r(TF^r U)$ then since $<,>$ is a global quantity $<\delta v_i, v_k>_x = <Dg(x)(\delta v_i), Dg(x)(v_k)>_{g(x)}$
So $TF^r g$ as above is the required diffeomorphism.

q.e.d.

<u>Proposition A3</u>. $F^r M$ is a Riemannian manifold.

Proof. Consider charts $(F^r U, F^r\varphi)$ for $F^r M$ and $(TF^r U, TF^r\varphi)$ for $TF^r M$ expressed in the above covariant co-ordinates. On each fibre $T_z F^r M$, $z = (x, v_1, \ldots v_r)$, the Riemann metric of M, g, induces an inner product and a norm given by:

$$\ll (\delta x \oplus \delta v_1 \ldots \oplus \delta v_r), (\delta x' \oplus \delta v_1' \ldots \oplus \delta v_r') \gg$$
$$= <\delta x, \delta x'>_x + <\delta v_1', \delta v_1>_x + \ldots\ldots + <\delta v_r, \delta v_r'>_x$$

and

$$|||(\delta x \oplus \delta v_1 \oplus \ldots \delta v_r)|||^2 = ||\delta x||_x^2 + ||\delta v_1||_x^2 + \ldots\ldots + ||\delta v_r||_x^2.$$

By Proposition A2 above these local expressions on $F^r U$, $TF^r U$ extend to global quantities since

$$<v, w>_x = <Dg(x)(v), Dg(x)(w)>_{g(x)} \quad \text{on the overlap}$$

of charts.

q.e.d.

The quadratic form describing the above is

$$d\sigma^2 = g_{ij}dx^i dx^j + g_{ij}\delta v_1^i \delta v_1^j + \ldots + g_{ij}\delta v_r^i \delta v_r^j.$$

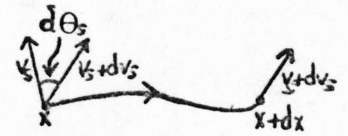
As before putting $\{x^I; I=1..(r+1)n\} = \{(x^1, \ldots\ldots, v_r^n)\}$, then direct substitution yields: for $d\sigma^2 = G_{IJ}\,dx^I\,dx^J$

<u>Proposition A4</u>. G has matrix forms:

$$G_{IJ} = \begin{array}{c c}
\left[\begin{array}{cccc}
g_{ij} + g_{mq}\Gamma^m_{ai}\Gamma^q_{bj}(\sum_t v^a_t v^b_t) & [pi,j]\,v_1^p & \cdots & [pi,j]\,v_r^p \\
[pj,i]\,v_1^p & g_{ij} & \cdots & 0 \\
[pj,i]\,v_2^p & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
[pj,i]\,v_r^p & 0 & \cdots & g_{ij}
\end{array}\right] &
\begin{array}{l} I=i \\ I=n+i \\ I=2n+i \\ \\ I=rn+i \end{array} \\
\begin{array}{cccc} J=j & J=n+j & \qquad & J=rn+j \end{array} &
\end{array}$$

$$G^{IJ} = \begin{array}{c c}
\left[\begin{array}{cccc}
g^{ij} & -\Gamma^j_{\mu l}\,g^{il}v_1 & \cdots & -\Gamma^j_{\mu l}\,g^{il}v_r \\
-\Gamma^i_{\mu l}\,g^{jl}v_1^\mu & g^{ij} + g^{\beta\gamma}\Gamma^j_{\beta\mu}\Gamma^i_{\gamma\tau}v_1^\mu v_1^\tau & \cdots & g^{\beta\gamma}\Gamma^j_{\beta\mu}\Gamma^i_{\gamma\nu}v_1^\mu v_r^\nu \\
-\Gamma^i_{\mu l}\,g^{jl}v_2^\mu & g^{\beta\gamma}\Gamma^j_{\delta\beta}\Gamma^i_{\epsilon\gamma}v_2^\delta v_1^\epsilon & \cdots & g^{\beta\gamma}\Gamma^j_{\delta\gamma}\Gamma^i_{\epsilon\beta}v_2^\delta v_r^\epsilon \\
\vdots & \vdots & \ddots & \vdots \\
-\Gamma^i_{\mu l}\,g^{jl}v_r^\mu & g^{\beta\gamma}\Gamma^j_{\delta\beta}\Gamma^i_{\epsilon\gamma}v_r^\delta v_1^\epsilon & \cdots & g^{ij} + g^{\beta\gamma}\Gamma^j_{\beta\delta}\Gamma^i_{\gamma\epsilon}v_r^\delta v_r^\epsilon
\end{array}\right] &
\begin{array}{l} I=i \\ I=n+i \\ I=2n+i \\ \\ I=rn+i \end{array}
\end{array}$$

Surprisingly this expression has a simple geometric interpretation. If we take two 'infinitesimally' close r-frames, $(x, v_1, \ldots v_r)$ and $(x + dx, v_1 + dv_1, \ldots, v_r + dv_r)$ say. then $ds^2 = g_{ij}dx^i dx^j$ measures the distance apart of the carriers, x and x+dx in the manifold M. Now suppose we parallel translate the vector $v_i + dv_i$ from $x_i + dx^i$ along the geodesic to x and let $d\theta_i$ be the angle that the translated vector makes with v_i at x



Then the distance between v_i and $v_i + dv_i$ can be taken to be $v_i^2 d\theta_i^2 = d\theta_i^2$ since $\|v_i\| = 1$.

Hence $d\sigma^2 = ds^2 + d\theta_1^2 + \ldots\ldots + d\theta_r^2$

. . . .

r-th order Equations on M, $r<n$.

First order equations on manifolds are vector fields on M, while Second order equations are vector fields on TM, X say, that satisfy the extra condition $T\pi_1 \circ X = id$. We could define r+1 order equations by considering certain submanifolds H^rM of T^rM and looking at certain vector fields on them. However we feel that the following approach using the Frame Bundles is much simpler, though not so general.

Definition: Let $p: F^{r+1}M \to M : (x, v_1, \ldots v_{r+1}) \to (x)$ be the projection.

Let $p_2: F^{r+1}M \to T^1M : (x, v_1, \ldots v_{r+1}) \to (x, v_1)$. similarly

Definition: An r-th order equation on M is a vector field, X, on $F^{r+1}M$ such that $Tp \circ X = p_2$.

Proposition A5. Locally X is of the form

$$X: (x, v_1, \ldots v_{r+1}) \to ((x, v_1 \ldots, v_{r+1})(v_1 \oplus \delta v_1 \oplus \ldots \oplus \delta v_{r+1}))$$

Proof:

$$Tp : TF^{r+1}M \to TM : ((x,v_1,\ldots,v_{r+1})(\delta x \oplus \delta v_1 \oplus \ldots \oplus \delta v_{r+1})) \to (x, \delta x)$$

q.e.d.

Definition: If $c : I \to F^{r+1}M$ is an integral curve of X, then $(p\circ c) : I \to M$ is a <u>Base Integral Curve</u> (Sub-flow line) of X.

<u>Proposition A6</u>: If $X : F^{r+1}M \to TF^{r+1}M$ is an r-th order equation on M of the form $X : (x,v_1,\ldots,v_{r+1}) \to ((x,v_1,\ldots v_{r+1})(v \oplus X_1 \oplus \ldots \oplus X_{r+1}))$, then any base integral curve $s : I \to M$ satisfies;

$$\dot{s}(t) = v(t), \nabla_{v(t)}v = X_1, \nabla_{v(t)}X_1 = X_2, \ldots \nabla_{v(t)}X_r = X_{r+1}$$

Proof. $s = p\circ c$ for some integral curve c of X.
Hence $\dot{s}(t) = T(p\circ c)(t,1) = Tp \circ Tc(t,1) = Tp(\dot{c}(t)) = Tp((X(t)) = v(t)$.
By definition of a vector field as the globalisation of local first order equations on a manifold in the standard co-ordinates then $X : (x,v_1,\ldots,v_{r+1}) \to ((x,v_1\ldots,v_{r+1})(\dot{x},\dot{v}_1,\ldots\dot{v}_{r+1}))$.
So in covariant co-ordinates:

$$X : (x,v_1,\ldots,v_{r+1}) \to ((x,v_1,\ldots,v_{r+1})(\dot{x} \oplus \dot{v}_1 + \Gamma(x)(v_1,\dot{x}) \oplus \ldots))$$
$$\to ((x,v_1,\ldots,v_{r+1})(\dot{x} \oplus \nabla_{\dot{x}(t)}v_1, \ldots \oplus \nabla_{\dot{x}(t)}v_{r+1})).$$

q.e.d.

Usually we shall require that the integral curves be defined for all time. i.e. if $c : I \to F^{r+1}M$ is an integral curve of X then c is defined over the range $(-\infty,\infty)$. If all the flow lines of X are so defined then the flow is <u>Complete</u>.

<u>Proposition A7</u>: If M is a compact manifold then any r-th order equation on M is complete.

Proof. It is known that a vector field on a compact manifold is complete. Abraham |1|. M compact implies that $F^{r+1}M$ is compact since $F^{r+1}M = \{(x,v_1,\ldots,v_{r+1}) : \|v_i\|_x = 1\}$.

q.e.d.

. . . .

Section 3; Frame Flows.

The geodesic flow on the set of r-frames ($r\leq n$) on a Riemannian manifold has long been studied and used as an example in many fields. This flow is defined by looking at the geodesics of the manifold, the motion of a particular r-frame then being the parallel translation of that r-frame along the geodesic determined by the leading element of the frame. We should like to investigate a more general situation in which the frames move along lines of known geodesic curvature in the manifold. The only known reference for this material is Arnold, |4|, where some results are stated without proof, and the examples are generally on the Lobachevsky Plane. In particular we shall give a differential geometric proof of the fact that these flows are incompressible (volume preserving).

Definition. A flow on F^rM, ($r\leq n$), is Isotropic if the sub-flow lines in M have constant speed. i.e. $\|(p\circ\dot{\alpha}(t))\| =$ constant.

Suppose we are given an r-tuple of non-negative numbers $\underline{k}=(k_1,\ldots,k_r)$ such that if $k_i=0$ then $k_{i+j}=0$ for all $j\geq 0$ i.e. $\underline{k}=(k_1,\ldots k_{i-1},0,\ldots 0)$ not $(k_1,0,k_3,\ldots k_r)$. then we should like to know if we can find a flow on F^rM based on lines of geodesic curvatures $\underline{k}$, which we shall term a $\underline{k}$-flow (on $F^{r+1}M$).

Definition. Given $\underline{k}$ as above then a $\underline{k}$-flow on $F^{r+1}M$ is a flow whose sub flow lines are lines of geodesic curvatures $\underline{k}$ and geodesic normals given by the last r elements of the r+1 frame.

<u>Proposition A8</u>: Given $\underline{k}$ as above then there is a unique $\underline{k}$ flow on $F^{r+1}M$.

Proof. Consider a chart $(F^{r+1}U, F^{r+1}\phi)$ for $F^{r+1}M$ and the induced covariant chart for $TF^{r+1}M$. Then

$X_{\underline{k}} : F^{r+1}U \rightarrow TF^{r+1}U.$

$:(x,v,w_1,\ldots,w_r) \rightarrow ((x,\ldots,w_r)(v \oplus k_1 w_1 \oplus \ldots \oplus k_{i+1}w_{i+1} - k_i w_{i-1} \oplus \ldots \oplus -k_r w_{r-1})$

is the local expression for a vector field on $F^{r+1}M$.

This follows since $k_{i+1}w_{i+1} - k_i w_{i-1} \in T_xU$ for all i and so if (V,Ψ) is another chart of M s.t. $U \cap V \neq \emptyset$, then the local expression for $X_{\underline{k}}$ transforms correctly on $F^{r+1}U \cap F^{r+1}V$. (Prop. A2). To complete the proof we must check that

$< \delta w_i, w_j > + < \delta w_j, w_i > = \delta^i_j$. (Lemma A1.).

(i) $<\delta v, v> = <k_1 w_1, v> = 0.$

(ii) $< w_i, \delta w_i > = < k_{i+1}w_{i+1} - k_i w_{i-1}, w_i > = 0.$

(iii) $<\delta v, w_1> + <\delta w_1, v> = <k_1 w_1, w_1> + <k_2 w_2 - k_1 v, v> = k_1 - k_1 = 0.$

(iv) $<\delta w_1, w_2> + <\delta w_2, w_1> = <k_2 w_2 - k_1 v, w_2> - <w_1, k_3 w_3 - k_2 w_1> = k_2 - k_2 = 0.$

$<\delta w_1, w_i> + <\delta w_i, w_1> = < k_2 w_2 - k_1 v, w_i> - <w_1, k_{i+1} w_{i+1} - k_{i-1} w_i> = 0.$

for $i > 2$

(v) $<\delta w_i, w_j> + <\delta w_j, w_i> = <k_{i+1}w_{i+1} - k_i w_{i-1}, w_j> + <k_{j+1}w_{j+1} - k_j w_{j-1}, w_i>$

$$= \begin{cases} 0 \text{ if } i=j \\ -k_{j+1} + k_{j+1} = 0 \quad \text{if } i = j+1 \\ k_j - k_j = 0 \quad \text{if } j = i+1. \\ 0 \quad \text{if } i \neq j-1, j, j+1. \end{cases}$$

So $X_{\underline{k}}$ is a vector field on $F^{r+1}M$ and has integral curves on $F^{r+1}M$, which by Propn. A6 project to sub flow lines $s : I \rightarrow M$ satisfying $\dot{s} = v, \nabla_v v = w_1, \ldots, \nabla_v w_i = k_{i+1}w_{i+1} - k_i w_{i-1}, \ldots$ etc.

i.e. s is a line of geodesic curvatures $\underline{k}$ and normals given by the last elements of the frame.

Conversely any flow in $F^{r+1}M$ generated by lines of geodesic curvature $\underline{k}$ give the vector field $X_{\underline{k}}$.

q.e.d.

Corollary A6:1. Given $m \in M$ and an initial direction $v \in T_mM$ and an r-tuple of normals $(N_1, \ldots, N_r)$ to v at m then there is a unique line of geodesic curvatures $\underline{k}$ passing through m with initial velocity v and geodesic normals at v, $(N_1, \ldots, N_r)$.

Remarks. (1). In the above we had $\underline{k}$ a constant but the same analysis as above would go through with $\underline{k} = \underline{k}(x)$, giving lines whose geodesic curvature depended on the position on the manifold, an isotropic flow.

(2) Suppose $\underline{k} = (k_1, \ldots k_s, 0, \ldots, 0)$, then the sub flow lines of $X_{\underline{k}}$ are lines of geodesic curvatures $k_1, \ldots, k_s$; geodesic normals $(w_1, \ldots, w_s)$ and the normals $(w_{s+1}, \ldots, w_r)$ are parallel translated along the sub flow lines, exactly as we obtained frames of reference along $\underline{k}$-lines in M in Section 1.

(3) We have $\underline{k}$-flows on $F^{r+1}M$ for all $r \leq n-1$. A $\underline{k}$-flow on $F^{r+1}M$ can be embedded into a $\underline{k}' = (k_1, \ldots, k_r, 0)$ flow on $F^{r+2}M$. So in effect the most general $\underline{k}$-flow we can obtain is on F^nM by taking any r-tuple of positive numbers $k_1, \ldots, k_r$ say, and looking at the resultant $\underline{k}' = (k_1, \ldots, k_r, 0, \ldots 0)$ flow.

We should now like to prove the following result using only the differential geometry of the frame bundles developed previously:

Theorem A7: Any $\underline{k}$-flow on $F^{r+1}M$, $r \leq n-1$, is incompressible

Recall that the Riemannian metric on $F^{r+1}M$ given by

$$\langle\langle v_1 \oplus \ldots \oplus v_{r+1}, w_1 \oplus \ldots w_{r+1} \rangle\rangle = \langle v_1, w_1 \rangle + \ldots \ldots + \langle v_{r+1}, w_{r+1} \rangle$$

is induced from the one on $TM \oplus \ldots\ldots \oplus TM$, r+1 times. Hence we can consider a '$\underline{k}$-flow' on $\oplus$ TM given by the vector field

$$X_{\underline{k}} : \oplus TM \longrightarrow T(\oplus TM)$$

$$: ((x,v)\oplus(x,w_1)\oplus \dots \oplus(x,w_r)) \longmapsto$$

$$((x,v,\dots w_r)(v\oplus k_1 w_1 \oplus \dots \oplus k_{i+1}w_{i+1} - k_i w_{i-1} \oplus \dots - k_r w_{r-1}))$$

In particular $X_{\underline{k}} / \oplus F^{r+1}M$ generates the $\underline{k}$-flow on $F^{r+1}M$.

So if $x^I = (x^1, \dots x^n, v^1_1, \dots \dots w^1_1 \dots \dots w^n_r)$ are the local co-ordinates then:

<u>Lemma A8</u>: The flow $X_{\underline{k}}$ is incompressible with respect to the Riemannian volume element on $TM \oplus \dots \oplus TM$ (r-times) iff the divergence of $X_{\underline{k}}$, namely $X^A{}_{,A} = 0$.

Proof: (Sasaki [23])

From this point we fix $\underline{k}$ and for ease of notation set $X_{\underline{k}} = X$.

<u>Lemma A9.</u> If G is the Riemann metric of $\oplus TM$ then:

(i) $\dfrac{\partial G_{JK}}{\partial x^i} G^{JK} = 2(r+2)\Gamma^h_{hi}$ (ii) $\dfrac{\partial G_{JK}}{\partial x^A} G^{JK} = 0 \quad n+1 \leq A \leq n(r+1)$.

<u>Sublemma</u> $v^\mu_s v^\nu_s \left(\partial/\partial x^i \left(g_{\beta\gamma}\Gamma^\beta_{\mu j}\Gamma^\gamma_{\nu k}\right)\right) g^{jk} + v^\mu_s v^\nu_s \dfrac{\partial g_{\beta\gamma}}{\partial x^i} g^{kl}\Gamma^\beta_{\mu k}\Gamma^\gamma_{\nu l} = 2\,\partial/\partial x^i [\lambda\nu, h]\Gamma^h_{\nu k} g^{jk} v^\lambda_s v^\nu_s$

Proof. Computation!

Proof of lemma.

G_{JK} and G^{JK} are given in the second section.

$$\begin{aligned}\frac{\partial G_{JK}}{\partial x^i} G^{JK} = {} & \partial/\partial x^i \left(g_{jk} + g_{mq}\Gamma^q_{lj}\Gamma^m_{pk}(v^p v^l + w_1^p w_1^l + \dots + w_r^p w_r^l)\right) g^{jk} \\ & -2\left(\partial/\partial x^i [\lambda j, k] v^\lambda\right)\Gamma^k_{\mu p} g^{jp} v^\mu - \dots - 2\left(\partial/\partial x^i [\lambda j, k] w_r^\lambda\right)\Gamma^k_{\mu p} g^{jp} v^\mu - \dots \\ & + \left(\frac{\partial g_{jk}}{\partial x^i}\right)\left(g^{jk} + g^{\beta\gamma}\Gamma^j_{\mu\beta}\Gamma^k_{\nu\gamma} v^\mu v^\gamma\right) - \dots + \left(\frac{\partial g_{jk}}{\partial x^i}\right)\left(g^{jk} + g^{\beta\gamma}\Gamma^j_{\mu\beta}\Gamma^k_{\nu\gamma} w_r^\mu w_r^\gamma\right)\end{aligned}$$

$$= \frac{\partial g_{jk}}{\partial x^i} \cdot g^{jk} + - - - + \frac{\partial g_{jk}}{\partial x^i} \cdot g^{jk} \ (r+2 \text{ times}) +$$

$$\partial/\partial x^i (g_{mq} \Gamma^q_{lj} \Gamma^m_{pq} v^l v^p) g^{jk} + \frac{\partial g_{jk}}{\partial x^i} \cdot g^{\beta\gamma} \Gamma^j_{\mu\beta} \Gamma^k_{\nu\gamma} v^\mu v^\nu - 2 (\partial/\partial x^i [\lambda j, k] v^\lambda) \Gamma^k_{\mu l} g^{jl} v^\mu +$$

$$\vdots$$

$$\partial/\partial x^i (g_{mq} \Gamma^q_{lj} \Gamma^m_{pq} w^l_r w^p_r) g^{jk} + \frac{\partial g_{jk}}{\partial x^i} g^{\beta\gamma} \Gamma^j_{\mu\beta} \Gamma^k_{\nu\gamma} w^\mu_r w^\nu_r - 2 (\partial/\partial x^i [\lambda j, k] w^\lambda_r) \Gamma^k_{\mu l} g^{jl} w^\mu_r .$$

$$= (r+2) \frac{\partial g_{jk}}{\partial x^i} \cdot g^{jk} \qquad \text{by sublemma.}$$

Now $\Gamma^h_{hi} = \frac{1}{2} g^{hp} (\cancel{g_{pi,h}} + g_{ph,i} - \cancel{g_{hi,p}}) = \frac{1}{2} g^{hp} g_{hp,i}$

$$\frac{\partial G_{JK}}{\partial w^i_s} G^{JK} = \partial/\partial w^i_s (g_{jk} + g_{mq} \Gamma^q_{lj} \Gamma^m_{pk} (v^l v^p - - + w^l_r w^p_r)) g^{jk} -$$

$$2 (\partial/\partial w^i_s [\lambda j, h] v^\lambda) \Gamma^h_{\nu k} g^{jk} v^\nu - - - - 2 (\partial/\partial [\lambda j, h] w^\lambda_r) \Gamma^h_{\nu k} g^{jk} w^\nu_r +$$

$$- - - + \cancel{\left(\frac{\partial g_{jk}}{\partial w^i_s}\right)} (g_{jk} + g^{\beta\gamma} \Gamma^k_{\nu\gamma} w^\mu_r w^\gamma)$$

$$= 2 g_{mq} \Gamma^p_{ij} \Gamma^m_{pk} w^p_s g^{jk} - 2 [ij, h] \Gamma^h_{\nu k} g^{jk} w^\nu_s$$

$$= 2 [ij, m] \Gamma^m_{pq} w^p_s g^{jk} - 2 [ij, h] \Gamma^h_{\nu k} g^{jk} w^\nu_s$$

$$= 0$$

Proposition A10. The Geodesic flow on $\oplus TM$ is incompressible.

Proof. $X : (x,v,w_1,\ldots w_r) \longmapsto ((x,v,\ldots w_r)(v\oplus 0 \oplus \ldots \oplus 0))$

$= ((x,v,..w_r)(v,-\Gamma(v,v),\ldots\ldots, -\Gamma(v,w_r))$. in standard co-ords.

Hence $X^i = v^i$, $X^{n+i} = -\Gamma^i_{jk}v^jv^k, \ldots\ldots, X^{rn+i} = -\Gamma^i_{jk}v^jv^k$.

$$X^I_{,I} = \frac{\partial X^I}{\partial x^I} + \Gamma^I_{JI}\; X^J, \text{ where } \Gamma \text{ is the Christoffel symbol for G.}$$

$$= \frac{\partial X^I}{\partial x^I} + \tfrac{1}{2}G^{IK}(G_{IK,J} + G_{JK,I} + G_{IJ,K})X^J$$

$$= \frac{\partial X^I}{\partial x^I} + \tfrac{1}{2}G^{IK}G_{IK.J}X^I \quad \because \text{ of summation over K,I.}$$

$$= \frac{\partial X^I}{\partial x^I} + \tfrac{1}{2}G^{JK}G_{JK.I}X^I$$

$$= \frac{\partial v^i}{\partial x^i} + \frac{\partial}{\partial v^i}(\Gamma^i_{pq}v^pv^q) + \ldots\ldots + \frac{\partial}{\partial w^i_r}(\Gamma^i_{pq}v^pw^q_r) + (r+2)\Gamma^h_{hi}.$$

$$= -2\Gamma^i_{iq}v^q - \Gamma^i_{iq}v^q\ldots - \Gamma^i_{iq}v^q + (r+2)\Gamma^h_{hi}v^i = 0.$$

q.e.d.

Proposition A11. Given $\underline{k} = (k_1,\ldots k_r)$, then the $\underline{k}$-flow on $\oplus TM$ is incompressible.

Proof.

$X(x,v,w_1,\ldots w_r) = (x,\ldots w_r)(v\oplus k_1w_1\oplus\ldots k_{i+1}w_{i+1}-k_iw_{i-1}\oplus\ldots\oplus -k_rw_{r-1})$

Changing to covariant co-ordinates we see that:

$X^i = v^i$, $X^{n+i} = k_1w_1 - \Gamma^i_{jk}v^jv^k$, $X^{pn+i} = k_pw_p - k_{p-1}w_{p-2} - \Gamma^i_{jk}v^jw^k_{p-1}\ \ldots\ldots$

$$\text{So } X^I_{,I} = \frac{\partial X^I}{\partial x^I} + \Gamma^I_{IJ}X^J$$

$$= \frac{\partial v^i}{\partial x^i} + k_1\frac{\partial w^i_1}{\partial v^i}\ \ldots + \frac{\partial(k_{j+1}w^i_{j+1} - k_jw^i_{j-1})}{\partial w^i_j}\ \ldots + k_r\frac{\partial w^i_{r-1}}{\partial w^i_r}.$$

$= 0.$ using the proposition above.

q.e.d.

We want to consider the $\underline{k}$-flow, not on $\oplus TM$ but on $F^{r+1}M$

To do this consider the following inclusions:-

$F^{r+1}M \longrightarrow F^rM \oplus T^1M \rightarrow F^{r-1}M \oplus T^1M \oplus T^1M \longrightarrow T^1M \oplus T^1M \oplus \ldots \oplus T^1M \rightarrow$
$T^1M \oplus \ldots \oplus T^1M \oplus TM \longrightarrow T^1M \oplus TM \oplus \ldots \oplus TM \longrightarrow TM \oplus TM \oplus \ldots \oplus TM.$

i.e. $M_s \hookrightarrow M_{s-1} \hookrightarrow M_{s-2} \hookrightarrow \ldots \hookrightarrow M_1 \hookrightarrow M_0$. say for $s=2r+1$ and M_j M_{j-1} as a codimension 1 submanifold.

We have the associated (Gauss) normal map for each inclusion $N_i : M_i \rightarrow TM_{i-1}$ s.t. $\langle N_i(m), v\rangle \neq 0$ for all $m \in M_i, v \in TM_{i-1}$

The Riemannian metric of $F^{r+1}M$ is induced from that on $TM \oplus \ldots \oplus TM$, and if $x^a = (, \ldots ,)$ is a co-ordinate patch on $F^{r+1}M$ induced from $x^A = (, \ldots ,)$ on $TM \oplus \ldots \oplus TM$, then the flow is incompressible on $F^{r+1}M$ iff $X^a_{,a} = 0$, and hence we need someway of relating $X^a_{,a}$ and $X^A_{,A}$. Hence we generalise the result of Sasaki [23]:

Lemma A 12. If $M_{i-1} \hookrightarrow M_i \hookrightarrow M_{i+1}$ are codimension one submanifold inclusions of Riemann manifolds, M_j, with unit normals N_i, N_{i+1}, and X is a vectorfield on M_{i-1} such that X/M_i and X/M_{i+1} are vectorfields on M_i, M_{i+1} resp., then

$$\operatorname{div}_{\Omega_{i+1}}(X/M_{i+1}) = X^A{}_{,A} - X_{B,C}\,(N^B_{i+1}N^C_{i+1} + N^B_iN^C_i), \text{ where}$$

x^A is the co-ordinate chart on M_{i-1} and Ω_{i+1} the induced Riemann volume element on M_{i+1}.

Proof.

Let (U,φ) be a chart of M_{i-1} with co-ordinates $\{x^A\}$, $A = 1,..n$.
(U,φ) the induced chart of M_i with co-ords $\{x^a\}$, $a = 1,..n-1$.
(U,φ) " " " " M_{i+1} " " $\{x^\alpha\}$, $= 1,..n-2$.
then

$$\operatorname{div}\Omega_{i+1}(X/M_{i+1}) = X^\alpha{}_{,\alpha}.$$

From the original lemma in Sasaki then

$$X^\alpha{}_{,\alpha} = X^a{}_{,a} - X_{b,c}N^b_{i+1}N^c_{i+1}.$$

So $X^{\alpha}_{,\alpha} = X^{A}_{,A} - X_{B,C}N^{B}_{i}N^{C}_{i} - X_{b,c}N^{b}_{i+1}N^{c}_{i+1}$.

Now $X_{b,c} = X_{B,C}\, e^{B}_{b} e^{C}_{c}$ where $e^{B}_{b} = \dfrac{\partial x^{B}}{\partial x^{b}}$ is the co-ordinate

change between TU/M_{i-1} and TU/M_{i}.

So $X^{\alpha}_{,\alpha} = X^{A}_{,A} - X_{B,C}(N^{B}_{i}N^{C}_{i} - e^{B}_{b}\, e^{C}_{c}\, N^{B}_{i+1}N^{C}_{i+1})$

$$= X^{A}_{,A} - X_{B,C}N^{B}_{i+1}N^{C}_{i+1} - X_{B,C}N^{B}_{i}N^{C}_{i}$$

q.e.d.

This generalises to:

Proposition A13. Given a family of codimension one inclusions of Riemann manifolds $M_s \hookrightarrow M_{s-1} \hookrightarrow \dots \hookrightarrow M_0$, with unit normals $N_i : M_i \longrightarrow TM_{i-1}$, then if X is a vectorfield on M_0 such that X/M_i is a vectorfield on M_i for each i then

$$\mathrm{div}_{\Omega_s}(X/M_s) = X^{A}_{,A} - X_{B,C}(N^{B}_{1}N^{C}_{1} + N^{B}_{2}N^{C}_{2} + \dots + N^{B}_{s}N^{C}_{s}),$$

where Ω_s is the induced volume on M_s and (x^{A}) are the co-ordinate charts of M_0.

Proof. By induction, since the lemma above gives the result for s=2. So assume true for s=k.

For the chart (U,φ) on M_0 let $\{x^{A}\}$, $A=1,\dots,n$. be the co-ords.

" " on M_1 let $\{x^{a}\}$, $a=1,\dots,n-1$ " " "

" " on M_2 let $\{x^{a}\}$, $a=1,\dots n-2$ " " "

⋮

" " on M_{k+1} let $\{x^{a_{k+1}}\}$, $a=1,\dots(n-k-1)$ " " "

By the Sasaki lemma

$$X^{a_{k+1}}_{,a_{k+1}} = X^{a_k}_{,a_k} - X_{b_k,c_k}(N^{b_k}_{k+1}N^{c_k}_{k+1})$$

$$= X^{A}_{,A} - X_{B,C}(\quad) - X_{b_k,c_k} N^{b_k}_{k+1} N^{c_k}_{k+1}$$

$$= X^{A}_{,A} - X_{B,C}(\quad) - X_{b_{k-1},c_{k-1}} e^{b_{k-1}}_{b_k} e^{c_{k-1}}_{c_k} N^{b_k}_{k+1} N^{c_k}_{k+1}$$

$$= X^{A}_{,A} - X_{B,C}(\quad) - X_{b_{k-1},c_{k-1}} N^{b_{k-1}}_{k+1} N^{c_{k-1}}_{k+1}$$

$$= X^{A}_{,A} - X_{B,C}(\quad) - X_{b_{k-2},c_{k-2}} e^{b_{k-2}}_{b_{k-1}} e^{c_{k-2}}_{c_{k-1}} N^{b_{k-2}}_{k+1} N^{c_{k-2}}_{k+1}$$

$$\vdots$$

etc

$$\vdots$$

$$= X^{A}_{,A} - X_{B,C}(\quad) - X_{B,C} N^{B}_{k+1} N^{C}_{k+1}$$

$$= X^{A}_{,A} - X_{B,C}(N^{B}_{1}N^{C}_{1} + \dots + N^{B}_{k+1}N^{C}_{k+1})$$

q.e.d.

Hence we have to know the normal maps for the inclusions:

$$F^{r+1}M \hookrightarrow F^{r}M \oplus T^{1}M \hookrightarrow F^{2}M \oplus T^{1}M \oplus T^{1}M \hookrightarrow T^{1}M \oplus T^{1}M \oplus \dots \oplus T^{1}M \hookrightarrow T^{1}M \oplus \dots \oplus TM \hookrightarrow \dots \hookrightarrow TM \oplus \dots \oplus TM.$$

For example

$$N_1 : T^{1}M \oplus TM \oplus \dots \oplus TM \longrightarrow T(TM \oplus TM \oplus \dots \oplus TM)$$

$$(x,v,w_1,\dots,w_r) \longmapsto (x,\dots w_r)(0 \oplus v \oplus 0 \dots \oplus 0)$$

then $|||N_1|||^{2} = 1.$ and

$$\langle\langle N_1(x,\dots w_r),(x,v,\dots w_r)(\delta x \oplus \delta v \oplus \dots \oplus \delta w_r)\rangle\rangle = 0$$

for $(x,v,\dots w_r)(x \oplus v \oplus \dots \oplus w_r) \quad T(T^{1}M \oplus TM \oplus \dots \oplus TM)$

Set

$$N_s : T^{1}M \oplus \dots \oplus T^{1}M \oplus TM \oplus \dots \oplus TM \longrightarrow T(T^{1}M \oplus \dots \oplus T^{1}M \oplus TM \oplus \dots \oplus TM)$$

$$N_{r+s} : F^{s}M \oplus T^{1}M \oplus \dots \oplus T^{1}M \longrightarrow T(F^{s-1}M \oplus T^{1}M \oplus \dots \oplus T^{1}M)$$

for $s=1,\dots r.$

$$N_1 \ :(x,v,\dots w_r) \longmapsto (x,v,\dots w_r)(0 \oplus v \oplus 0 \oplus \dots \oplus 0)$$

$$N_2 \ :(\qquad) \longmapsto (\qquad)(0 \oplus 0 \oplus w_1 \oplus \dots \oplus 0)$$

$$\vdots$$

$$N_{r+1} :(\qquad) \longmapsto (\qquad)(0 \oplus 0 \oplus \dots \oplus w_r)$$

$N_{r+2}(\ ) = (\ \ )(0 \oplus \ldots\ldots \oplus 0 \oplus w_r \oplus w_{r-1})$

$N_{r+3}(\ ) = (\ \ )(0 \oplus \ldots\ldots \oplus w_{r-1} \oplus w_{r-2} \oplus 0)$

$\vdots$

$N_{2r+1}(\ ) = (\ \ )(0 \oplus w_1 \oplus v \oplus \ldots\ldots \oplus 0 \oplus 0)$

In addition we also need some information about the Christoffel symbols for G on $TM \oplus TM \oplus \ldots \oplus TM$, $[\![\ ,\ ]\!]$ and Γ say.

Lemma A14. $\Gamma^{I}_{pn+j\ pn+k} = [\![pn+j\ pn+k,\ I]\!] = 0$ for $I = 0,\ldots(r+1)n$,
$p,q = 1,\ldots r+1$.

Proof. $[\![JK,H]\!] = \frac{1}{2}(G_{JH,K} + G_{HK,J} - G_{JK,H})$ by definition

(i) $[\![pn+j\ pn+k, pn+h]\!] = \frac{1}{2}(G_{pn+j,pn+h,pn+k} + G_{pn+h\ pn+k,pn+j} - G_{pn+j\ pn+k,h+pn}) = 0$

(ii) $[\![pn+j\ pn+k, h]\!]$

$$= \frac{1}{2}\left(\frac{\partial}{\partial v^k_p}[ah,j]v^a_p + \frac{\partial}{\partial v^j_p}[ah,k]\,v^a_p - g_{jk,h}\right)$$

$$= \frac{1}{4}(g_{kj,h}+g_{hj,k}-g_{kh,j}+g_{jk,h}+g_{hk,j}-g_{jh,k}-2g_{jk,h}) = 0.$$

(iii) $[\![pn+j\ pn+k,\ qn+h]\!]$, $p \neq q$,

$$= \tfrac{1}{2}\left(\partial/\partial v^k_p(0) + \partial/\partial v^j_p(0) - \partial/\partial v^h_q(g_{jk}(x))\right) = 0.$$

(iv) $[\![pn+j\ qn+k,h]\!]$, $p\neq q$,

$$= \tfrac{1}{2}\left(\partial/\partial v^k_q([ah,j]\,v^a_q) + \partial/\partial v^j_p([ah,k]\,v^a_q) - \partial/\partial x^h(0)\right).$$

$= 0.$

(v) $[\![pn+j\ qn+k,pn+h]\!]$, $p \neq q$,

$$= \tfrac{1}{2}\left(\partial g_{jh}/\partial v^k_q\right) = 0.$$

(vi) $[\![pn+j\ qn+k,pn+h]\!]$, $p \neq q$,

$$= \tfrac{1}{2}\left(\partial g_{hk}/\partial v^j_p\right) = 0.$$

(vii) $[\![pn+j\ qn+k,rn+h]\!]$, $p \neq q \neq r$, $=0$.

So $[\![pn+j\ pn+k,\ I]\!] = 0$ for all I and $p,q \geqslant 1$, $jk=0,$--n.

Now $\Gamma^{I}_{JK} = \frac{1}{2} G^{IP} [\![JK,P]\!]$

So $\Gamma^{I}_{pn+j\ qn+k} = \frac{1}{2} G^{IP} [\![pn+j\ qn+k, P]\!] = 0.$

q.e.d.

Lemma A15. The geodesic flow on $TM \oplus TM \oplus \ldots \oplus TM$ has co-ords $X_{pn+i} = 0$ for $p > 0$.

Proof, Recall $X^{i} = v^{i}$, $X^{n+i} = -\Gamma^{i}_{pq} v^{p} v^{q}$, $X^{2n+i} = -\Gamma^{i}_{pq} v^{p} w^{q}_{1}$,

......., $X^{rn+i} = \Gamma^{i}_{pq} v^{p} w^{q}_{r}$.

Now $X_{A} = X^{B} G_{AB}$,

Hence

$$X_{n+i} = G_{n+i\,B} X^{B} = G_{n+i\,j} v^{j} + G_{n+i\,n+j}(-\Gamma^{j}_{pq} v^{p} v^{q}) + 0.X^{mn+j}.$$

$$= [aj,i]\, v^{a} v^{j} - g_{ij} \Gamma^{j}_{ab} v^{a} v^{b}$$

$$= [aj,i]\, v^{a} v^{j} - [ab,i]\, v^{a} v^{b} = 0$$

$$X_{mn+i} = G_{mn+i\,j} v^{j} + G_{mn+i\,mn+j}(-\Gamma^{j}_{ab} v^{a} v^{b})$$

$$= [aj,i]\, v^{j} w^{a}_{m-1} - g_{ij} \Gamma^{j}_{ab} v^{a} w^{b}_{m-1}.$$

$$= 0 \qquad \text{since} \quad G_{mn+j\,mn+k} = \Gamma^{m}_{p}\, g_{jk}.$$

q.e.d.

Theorem A16 The Geodesic flow on $F^{r+1}M$ is incompressible.

Proof . By Propn. A7, that any k-flow on $\oplus TM$ is incompressible, $X^{A}_{,A} = 0$; so sufficient to prove that $X_{B,C}(N^{B}_{1}N^{C}_{1} + \ldots + N^{B}_{s}N^{C}_{s}) = 0$

$$X_{B,C} = X_{B.C} + \Gamma^{A}_{BC} X_{A}$$

Then $N^{a}_{1} = \ldots\ldots = N^{a}_{s} = 0$ for $a=1,\ldots n.$ $s=1,\ldots(r+1)n.$

So $X_{B,C}(N^{B}_{1}N^{C}_{1} + \ldots + N^{B}_{s}N^{C}_{s}) = X_{P,Q}(N^{P}_{1}N^{Q}_{1} + \ldots\ldots + N^{P}_{s}N^{Q}_{s})$ for $P,Q = n+1\ldots$
$\ldots.(r+1)n.$

Now $\Gamma^{A}_{BC} = 0$ by Lemma A13, so $X_{P,Q} = X_{P.Q}$

However $X_{P} \equiv 0$ by Lemma above and so $X_{P.Q} = 0$.

q.e.d.

<u>Theorem A17.</u> Any <u>k</u>-flow on $F^{r+1}M$ is incompressible.

Proof. By above $X_{B,C} = X_{B.C}$ so again we can consider

$$X_{B.C}\,(N_1^B N_1^C \ldots\ldots + N_s^B N_s^C)$$

(i) Consider $X_{B.C}N_1^B N_1^C = \dot{X}_{n+i.n+j}v^i v^j = \partial/\partial v^j(g_{ip}k_1 w_1^p)v^i v^j = 0$

(ii) Consider $X_{B.C}N_p^B N_p^C$ for $p=2,\ldots r.$

$$= X_{pn+i.pn+j}w^i_{p-1}w^j_{p-1}$$

$$= \partial/\partial w^j_p(g_{iq}(k_{p+1}w^q_{p+1} - k_p w_{p-1}))w^i_{p-1}w^j_{p-1} = 0.$$

(iii) Consider $X_{B.C}N_p^B N_p^C$ for $p=r+1,\ldots,s.$

$$N_p = (x,v,\ldots,w_r)(0 \oplus \ldots \oplus w_p \oplus w_{p-1} \oplus \ldots \oplus 0)$$

$$X_{B.C}N_p^B N_p^C = X_{(p-1)n+j,(p-1)n+i}w^i_p w^j_p + X_{pn+j,pn+i}w_{p-1}w_{p-1} + X_{(p-1)n+j,pn+i}w^j_{p-1}w^i_p$$
$$+ X_{pn+j,(p-1)n+i}w^j_p w^i_{p-1}$$

$$= \partial/\partial w_{p-1}(k_p w^j_p - k_{p-1}w^j_{p-2})w^j_p w^i_p + \partial/\partial w_p(k_{p+1}w_{p+1} - k_p w_{p-1})w^i_{p-1}w^j_{p-1}$$
$$+ \partial/\partial w_p(k_p w^j_p - k_{p-2}w_{p-2})w^i_p w^j_p + \partial/\partial w_{p-1}(-k_p w_{p-1} + k_{p+1}w_{p+1})w^j_{p-1}w^i_p$$
$$= 0 + 0 + k_p w^i_p w^i_{p-1} - k_p w^i_{p-1}w^i_p = 0.$$

So $X_{B.C}(N_1^B N_1^C + \ldots + N_s^B N_s^C) = 0.$

q.e.d.

Comment. In some work on the geodesic flow it is considered on the bundle of ordered r-frames, not necessarily orthogonal; the same analysis on $T^1M \oplus \ldots \oplus T^1M$ will show that this flow is also incompressible.

APPENDIX B : Some concepts from Classical Mechanics.

The purpose of this section is to show that k-flows arise naturally in the modern treatment of Classical Mechanics, as laid out in Abraham([1]) and Maclane([19]).

Suppose we have a particle moving in the plane with velocity v; then we have Kinetic Energy $\frac{1}{2}mv^2$. More generally if we have a Riemannian manifold, M, with metric g, then we can assign to a particle at m, moving with velocity $v \in T_mM$ the Kinetic energy $\frac{1}{2}g(m)(v,v)$.

Definition: $T : TM \to \mathbb{R} : (x,v) \mapsto \frac{1}{2}g(x)(v,v)$.

Adding an electric field onto the plane can be interpreted as adding in a further (Potential) energy function $V : TM \to \mathbb{R}$ st V/T_mM is constant. ie depends only on the position on the manifold.

Definition: Given Kinetic and Potential Energy terms as above then $L = T - V$ is the Lagrangian.

(ii) Let $L_x = L/T_xM$ and DL_x the fibre derivative then $FL : TM \to T^*M : (x,v) \mapsto (x, DL_x(v)) \in L(T_xM, \mathbb{R})$ is the Legendre Transformation; called regular if FL is a local diffeomorphism, hyperregular if FL is a diffeomorphism.

(iii) $A : TM \to \mathbb{R} : (x,v) \mapsto FL(x,v).(x,v)$ is the Action.

(iv) $E : TM \to \mathbb{R}$ by $E = A - L$ is the Total Energy.

When we have T,V as above then $A = 2T$ and $E = T + V$.

It turns out that the cotangent bundle because of invariance properties is of more interest than the tangent bundle. In particular the Hamiltonian is a function on this space. $H : T^*M \longrightarrow \mathbb{R}$. Also T^*M has a natural volume element with respect to which divergence theorems can be stated, say Ω. If FL is a diffeo or a local diffeo then there is an induced volume on TM $\Omega_L = (FL)^*\Omega$, that is for each $v \in TM$ there is a map $\Omega_L(v) : T_vTM \times T_vTM \longrightarrow \mathbb{R}$ st if $\Omega_L(v)(\xi_1, \xi_2) = 0$ for all ξ_1 then $\xi_2 \equiv 0$.

Ω_L induces a map $\Omega_L^\flat : TTM \longrightarrow T^*TM$ by $\Omega_L^\flat(v)(\xi)(\eta) = \Omega_L(v)(\xi, \eta)$ and a map $\Omega_L^\# = (\Omega_L^\flat)^{-1}$; $T^*TM \longrightarrow TTM$.

We use this to define a vectorfield on TM, say X_E using the total energy $E : TM \longrightarrow \mathbb{R}$. $TE : TTM \longrightarrow \mathbb{R} \times \mathbb{R}$ so let $dE : TM \longrightarrow T^*TM : v \longmapsto P_2 \circ T_vE$, where P_2 projects to the second factor. We then obtain the vectorfield by

$$
\begin{tikzcd}
TTM & \\
TM \arrow[u, "X_E"] \arrow[r, "dE"'] & T^*TM \arrow[ul, "\Omega^\#_L"']
\end{tikzcd}
$$

This rather unnatural definition justifies its existence in the following:-

Proposition (Abraham [1]) If L is a regular Lagrangian with energy E then X_E is a second order equation on M with subflow lines $c : I \quad M$ satisfying:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)(c, \dot{c}) - \frac{\partial L}{\partial x^i}(c) = 0 \text{ , the usual Lagrangian}$$

equations, and $T\pi_1 \circ X_E = id$.

Proof: Abraham pp 120-122.

When $L = T - V$ then $E = T + V$ and FL is the natural isomorphism $j : TM \longrightarrow T^*M : (x,v) \longmapsto g(x)(v,v)$ $j = 2T$.

(i) If $V = 0$ then Abraham shows that the sub flow lines are geodesics of M with constant velocity, so choosing an energy level e and require $E = e$ (total energy of the system is conserved) then we get lines in the energy level e, corresponding to a particle on a plane moving under no external forces.

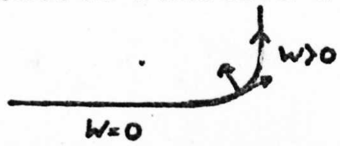
(ii) If $V \neq 0$ then choose an energy level e st $e > V(m)$ for all $m \in M$ then define $g_e = (e-V)\,g$ as a Riemann metric on M , the Jacobi metric.

Again choosing a fixed energy level e for the total energy $(\frac{1}{2}g_e + V)$ then Abraham shows that the trajectories are geodesics of g_e, which are the geodesics of g but reparameterised so the g_e velocity of the lines is constant. This corresponds to the motion of a charged particle on the plane moving under an Electric field which speeds the particles up, but keeps them travelling in a straight line.

This is sufficient to motivate a close examination of the geodesic flow on Riemann manifolds, since choosing an energy level, say velocity $\equiv 1$, then the geodesic flow on T^1M is a global description of all the possible orbits of the system.

The situation alters when we introduce a magnetic field onto the plane, when the flow lines are bent into arcs of circles, which are no longer the geodesics of $\mathbb{R}^2$ but k-lines. The radius of the circle depends on the velocity at the point so let us introduce a velocity dependent potential of the simplest form.

Definition: $W : TM \longrightarrow \mathbb{R}$ is a velocity potential if on on fibres it is of the form $W(x,v) = W^i(x)v^i$, ie. linear on fibres, for a smooth function $W(x)$.



Proposition B:1. If $L = T - V - W$ is a Lagrangian then $A = 2T - W$, $E = T + V$ and L is regular.

Proof: $L : TM \longrightarrow \mathbb{R} : (x^i, v^i) \longmapsto \frac{1}{2} g_{ij}(x)(v^i v^j) - V^i(x) - W^i(x)v^i$

$FL : (x^i, v^i) \longmapsto DL_x(v)$ the fibre derivative.

$DL_x(v) = \partial/\partial v^i (Lx)_v = g_{ij}(x)(v,-) - W^i(x)(-) - L(T_xM, \mathbb{R}) = T_xM$

$DL_x(v)(w) = g(x)(v,w) - W(x)w$ since V is constant on fibres.

So $FL(v)(w) = j(v,w) - W(x)w$ where j is the natural iso.

(i) $A : TM \longrightarrow \mathbb{R} : v \longmapsto FL(v)(v) = g(v,v) - W(x)v = 2T(v) - W(v)$

(ii) $E = A - L = (2T - W) - (T-V-A) = T + V$.

(iii) On fibres

FL is 1:1 for suppose $FL(u_1) = FL(u_2)$ then $FL(u_1)(w) = FL(u_2)(w)$ $j(u_1,w) = j(u_2,w)$ since W is constant on fibres. Hence $u_1 = u_2$ since j is a diffeomorphism.

FL is onto If $f_x \in T_x^*M$ then $f_x : T_xM$ $\mathbb{R}$ and $(W(x) + f_x) : T_xM \longrightarrow \mathbb{R}$. So $(W(x) + f_x) \in T_x^*M$ and hence there is v st $j(v) = (W(x) + f_x)$

So $FL(v) = j(v) - W(x) = f_x$

Hence we have an inverse function

$T_x^*M \longrightarrow T_xM : f_x \longmapsto j^{-1}(f_x + W_x)$ and is smooth since j is a diffeo and W is smooth by definition.

FL is a fibre preserving diffeo and hence L is regular.

q.e.d.

Notes: $E = T + V$ shows that W does not contribute to the total energy, as the velocity for the magnetic field is not increased only the paths are bent.

Choosing an energy level e st $V(m) > e$ for all m, then Abraham says that the sub flow lines satisfy Lagranges equations. Considering the Jacobi metric g_e let $\underline{T} : TM \to \mathbb{R} : (x,v) \longmapsto \frac{1}{2} g_e(x)(v,v)$ be the associated Kinetic energy.

<u>Lemma B:2.</u> If $c : I \to M$ is a sub flow line of X_E and $a : I \to M : s \mapsto a(s)$ a reparameterisation of the line using the arc length s of g_e, then the function $p : I \to I$, st $a(s) = c(p(s))$ ie $t = p(s)$ satisfies $p'(s) = 1/\sqrt{2}(e-V(a(s))$

Proof: $\underline{T}(a(s), a'(s)) = \underline{T}(c(p(s)), C'(p(s))\, p'(s))$

$$= (p'(s))^2\, \underline{T}(c(t), C'(t)) \text{ since } \underline{T} \;\; q^c \text{ in } v$$

$$= (p'(s))^2 (e-V(a(s)))\, T(c(t), c'(t)) \ldots (1)$$

since $\underline{T} = (e-V)T$. Choose an energy level e so $T + V = e$

so $\underline{T}(c(t), c'(t)) = T^2(c(t), c'(t))$ ....(2)

We are parameterising the curve a by arc length s where $ds^2 = (e-V) g_{ij} dx^i dx^j$, and so $g_e(a'(s), a'(s)) \equiv 1$.

So $\underline{T}(a(s), a'(s)) \equiv \frac{1}{2}$.

So by above (1) & (2)

$$\underline{T}(c(t), c'(t)) = (1/p'(s))^2\; \underline{T}(a(s), a'(s))$$

and hence $\underline{T}(c(t), c'(t)) = 1/2p'(s)^2\; T^2(c(t), c'(t))$ from (2))

So $\frac{1}{2} = (p'(s))^2 (e-V(a(s)) \cdot \dfrac{1}{\sqrt{2}\, p'(s)}$

q.e.d.

Lemma B:3. Given a velocity potential W and a Lagrangian $L = T - V - W$ then if $c : I \longrightarrow M$ is a subflow line of X_E parameterised by the arc length of the Jacobi metric, g_e, then letting $v(s) = \dot{c}(s)$:-

$$(\nabla_{v(s)} v)^p = \frac{1}{\sqrt{2}} (g_e)^{pk} \left\{ \frac{\partial W^k}{\partial x^j} v^j - \frac{\partial W^j v^j}{\partial x^k} \right\}$$

where ∇ is the compatible connexion for g_e.

Proof: c satisfies:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0 . \qquad \text{Lagranges equations.}$$

$$0 = \frac{d}{dt}\left(\frac{\partial T}{\partial v} - \frac{\partial W}{\partial v}\right) - \frac{\partial}{\partial x}(T - V - W)$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) - \frac{\partial}{\partial x}(T-V) - \frac{d}{dt}\left(\frac{\partial W}{\partial v}\right) - \frac{\partial W}{\partial x}$$

$$= A(x,v) - B(x,v) \quad \text{say.}$$

parameterising by arc length s then $\dfrac{d}{dt} = \dfrac{1}{p'(s)} \dfrac{d}{ds}$.

$$A = \frac{1}{p'(s)} \left\{ \frac{\partial T}{\partial v}\left(c(s), \frac{c'(s)}{p'(s)}\right) \right\} - \frac{\partial}{\partial x}\left(T\left(c(s), \frac{c'(s)}{p'(s)}\right)\right) - \frac{\partial V}{\partial x}(c(s))$$

$$= \frac{1}{p'} \frac{d}{ds} \left\{ \frac{1}{e-V} \frac{1}{p'} \frac{\partial \underline{T}}{\partial v}(c(s), c'(s)) \right\} - \frac{2}{p'} \frac{\partial \underline{T}}{\partial x}(c, c') -$$

$$\frac{\partial V}{\partial x} \left(\frac{\underline{T}}{(p')^2 (e-V)^2} - 1 \right)$$

using linearity of $\underline{T}$, and notation of previous lemma.

$$= \frac{2}{p'} \frac{d}{ds} \frac{\partial \underline{T}}{\partial v}(c, c') - \frac{\partial \underline{T}}{\partial x}(c, c')$$

since $\underline{T} \equiv \frac{1}{2}$ and $p' = (\sqrt{2}(e-V))^{-1}$

$\underline{T} = \frac{1}{2}(g_e)_{ij}v^i v^j$.

Hence $\frac{d}{ds}\left(\frac{\partial T}{\partial v^k}\right) - \frac{\partial T}{\partial x^k}$

$$= (g_e)_{ik}\dot{v}^i + \tfrac{1}{2}\left((g_e)_{ik,j} + (g_e)_{jk,i} - (g_e)_{ij,k}\right)v^i v^j$$

using chain rule and symmetry of g_e.

So $A_k = \frac{\sqrt{2}}{p'}\left((g_e)^{pk}(g_e)_{ki}\dot{v}^i + \frac{1}{2}(g_e)^{pk}((g_e)_{ik,j}+(g_e)_{jk,i}-(g_e)_{ij,k})\right)$

$= \frac{\sqrt{2}}{p'}\left(\dot{v}^p + \Gamma^p_{ij}\, v^i v^j\right)$ where are the Christoffel symbols of g_e....Spivak(25)

$= \frac{\sqrt{2}}{p'}\left(\nabla_v v\right)_p$

. . . . . .

$B_k = \frac{d}{dt}\left(\frac{\partial W}{\partial v^k}(c(t),\dot{c}(t))\right) - \frac{\partial W}{\partial x^k}(c(t),\dot{c}(t))$ since $W(c,\dot{c}) = W^j v^j$

$= \frac{1}{p'}\frac{d}{ds}\left(\frac{\partial}{\partial v^k}(W^i(c(s))\, v^i(c'(s)/p'(s))\right) - \frac{\partial W^j}{\partial x^k}(c(s))\, v^j(c'/p')$

$= \frac{1}{p'}\left\{ \frac{dW^k}{ds} - \frac{\partial W^j}{\partial x^k} v^j \right\}$

By Lagranges equations $A_k - B_k = 0$

So

$$\nabla_v v_p = \frac{1}{\sqrt{2}}(g_e)^{pk}\left\{ \frac{dW^k}{ds}(c(s)) - \frac{W^j}{x^k}(c(s))\, v^j(c,\dot{c}) \right\}$$

and the result follows by chain rule.

q.e.d.

Corollary B : 3 ; 1. If $W \equiv 0$ then if $L = T - V$ then the sub flow lines of X_E are the geodesics of g_e.

Proof: $W^k \equiv 0$.

q.e.d.

We can regard W as a covectorfield on M since for each $m \in M$ $W(m) : T_m M \longrightarrow \mathbb{R} : v \longmapsto W^i(m)v^i$.
ie, $W : M \longrightarrow T^* M$.

Definition(Willmore): If λ is a covectorfield with associated 1-form Φ, then curlλ is the skew-symmetric tensorfield with components the exterior derivative of Φ.

In our case $\lambda = W^i dx^i$ since $\Phi(m)v = W^i dx^i (v^j \partial / \partial x^j)$
$= W^i v^i$.

So $d\Phi = dW^i \wedge dx^i = \frac{\partial W^i}{\partial x^j} dx^j \wedge dx^i$

So $(\text{curl}\, W)_{ij} = \left(\frac{\partial W^i}{\partial x^j} - \frac{\partial W^j}{\partial x^i}\right)$

Let $(\text{curl}\, W)^i_j = (g_e)^{ip} (\text{curl}\, W)_{pj}$

Proposition B:4. Given a Lagrangian $L = T - V - W$ then given a subflow line of X_E parameterised by the arc-length s of g_e and putting $\dot{c}(s) = v(s)$,
then $\nabla_{v(s)} v = K(c, \dot{c})\, N(v(s))$, for a unit vector N normal to v and $K^2 = \frac{1}{2}(g_e)_{ij} (\text{curl}\, W)^j_p (\text{curl}\, W)^i_q v^p v^q$. ie. K^2 is the norm of $\frac{1}{\sqrt{2}}(\text{curl}\, W)^i_j v^j \partial / \partial x^i \in T_{c(s)} M$.

Proof: $\dfrac{dW^k}{ds} - \left(\dfrac{\partial W^j}{\partial x^k}\right)v^j = \dfrac{\partial W^k}{\partial x^j}\cdot\dfrac{dx^j}{ds} - \dfrac{\partial W^j}{\partial x^k}\cdot v^j$

$$= \left(\frac{\partial W^k}{\partial x^j} - \frac{\partial W^j}{\partial x^k}\right) v^j = \sum_j (\mathrm{curl}\, W)_{kj} v^j$$

So $[\nabla_v v]_p = \dfrac{1}{\sqrt{2}} (g_e)^{pk} (\mathrm{curl}\, W)_{kj} v^j = \dfrac{1}{\sqrt{2}} (\mathrm{curl}\, W)^p_j v^j = C^p$ say

Now C^p is perpendicular to v since:-

$$\langle C^p, v\rangle = \frac{1}{\sqrt{2}} (g_e)_{ij} (\mathrm{curl}\, W)^i_p v^p v^j$$

$$= \frac{1}{\sqrt{2}} (g_e)_{ij} (g_e)^{iq} (\mathrm{curl}\, W)_{pq} v^p v^q$$

$$= \frac{1}{\sqrt{2}} (\mathrm{curl}\, W)_{pq} v^p v^q$$

$$= \frac{1}{\sqrt{2}} \left(\frac{\partial W^p}{\partial x^q} - \frac{\partial W^q}{\partial x^p}\right) v^p v^q$$

$$= \frac{1}{\sqrt{2}} \sum_{p,q} \frac{\partial W^p}{\partial x^q} v^p v^q - \frac{1}{\sqrt{2}} \sum_{p,q} \frac{W^q}{x^p} v^p v^q = 0.$$

So $\nabla_v v = K(x,v)N(x,v)$ for some unit vector N perp to v.

where $K^2 = \frac{1}{2}(g_e)_{ij}(\mathrm{curl}\, W)^i_p (\mathrm{curl}\, W)^j_q v^p v^q = \frac{1}{2}\left\| (\mathrm{curl}\, W)^i_p v^p \partial/\partial x^i \right\|^2$

q.e.d.

Hence in the above the lines of X_E are lines of variable geodesic curvature.

Examples:

(i) If W can be expressed as a gradient field, ie there is $f : M \to \mathbb{R}$ st $W = df$ and $W_i = (\mathrm{grad}\, f)_i = \dfrac{\partial f}{\partial x^i}$

For the exterior derivative $d^2 = 0$ .So $0 = \mathrm{curl}\, W = \mathrm{curl}.\mathrm{grad}\, W$.

So the flow lines are geodesics of g_e.

(ii) The next simplest case is on $\mathbb{R}^{2n}$ where W has curl W represented by an orthogonal matrix, skew-symmetric. ie when $(\text{curl}\,W)_{ij} = 2K_o A_{ij}$ for constant K_o and $A = (A_{ij})$ a skew orthogonal matrix. Then $g_{ij} = \delta^i_j$ and so

$$(\text{curl}\,W)^i_j = (\text{curl}\,W)_{ij}.\ K^2 = \tfrac{1}{2}\, 2K_o Av, K_o Av = K_o^2\, A^T Av, v$$
$$= K_o^2 \langle v, v\rangle = K_o^2 .$$

since $\|v\| = 1$ by choice of energy surface.

This is precisely the description of the motion of a charged particle on the plane under the influence of a magnetic field H of constant strength $B = \|\text{curl}\,H\|$ and we take for example $A(x,y) = (By, (1-B)x)$. A proof of this using straight variational principles can be found in Anosov and Sinai ([3]).

(iii) Consider the Lobachevsky plane L^2 with the usual metric : $ds^2 = \dfrac{dx^2 + dy^2}{1-x^2-y^2} = \dfrac{dx^2+dy^2}{A^2}$ so $g_{ij} = \delta^i_j / A^2$ say.

Let us take $V \equiv 0$ and some positive number k.

Let $W(x,y) = \left(\dfrac{(2k)^{\frac{1}{2}}}{4(1-x^2)^{\frac{1}{2}}} \log\dfrac{(1-x^2)^{\frac{1}{2}}+y}{(1-x^2)^{\frac{1}{2}}-y}, \dfrac{-(2k)^{\frac{1}{2}}}{4(1-y^2)} \log\dfrac{(1-y^2)^{\frac{1}{2}}+x}{(1-y^2)^{\frac{1}{2}}-x}\right)$

$= (W_1, W_2)$ say.

$L((x,y),(v_1,v_2)) = \frac{1}{2}g(v,v) - W^i(x,y)v^i = T - W$ say

Choose an energy level $e = 1$, giving the Jacobi metric $g_e = g$.

$(\text{curl}\,W)_{11} = (\text{curl}\,W)_{22} = 0$

$(\text{curl}\,W)_{12} = -(\text{curl}\,W)_{21} = \dfrac{\partial W^1}{\partial y} - \dfrac{\partial W^2}{\partial x}$

So $(\operatorname{curl} W)_{ij} = \begin{bmatrix} 0 & \frac{\sqrt{2}\, k^{\frac{1}{2}}}{A^2} \\ \frac{-\sqrt{2}\, k^{\frac{1}{2}}}{A^2} & 0 \end{bmatrix}$

$$(\operatorname{curl} W)^i_j = A^2 \delta_{ip} (\operatorname{curl} W)_{pj} = A^2 (\operatorname{curl} W)_{ij}$$

$$(\operatorname{curl} W)^i_j = \begin{bmatrix} 0 & \sqrt{2}\, k^{\frac{1}{2}} \\ -\sqrt{2}\, k^{\frac{1}{2}} & 0 \end{bmatrix}$$

$$K^2 = \tfrac{1}{2} g_{ij} (\operatorname{curl} W)^i_p (\operatorname{curl} W)^j_q v^p v^q$$
$$= \tfrac{1}{2} A^2 (2k^2 v_2 v_2 + 2k^2 v_1 v_1) = \frac{k^2 \delta^i_j v^i v^j}{A^2}$$

$= k^2 g(v,v) = k^2.$ since $\|v\| \equiv 1$, parameterised by arclength.

So $\nabla_v v = k_1 N$ for some unit N.

Since we are on a surface $\nabla_v N = -kv$ since (v,N) is a 2-frame. The flow lines are thus the <u>k</u>-lines of L^2.

By the previous theorem $\nabla_v v^p = \frac{1}{\sqrt{2}} (\operatorname{curl} W)^p_j v^j$

implies that $[k^{\frac{1}{2}} N_1, k^{\frac{1}{2}} N_2] = \begin{bmatrix} 0 & k^{\frac{1}{2}} \\ -k^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

so $N(v_1, v_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

So the flowlines are the <u>k</u>-lines of L^2 with left hand normals.

This we feel motivates the study of the k-flows on T^1M, since if we are presented with a mechanical situation it is uaually describable by the Lagrangian case. So given the general case of a velocity potential the k-flow generated on T^1M globally describes the possible trajectories of a particle on the manifold given an initial velocity.

Symbols.

M, N	Riemann manifolds
$g, \langle\,\rangle, \|\ \|$	Riemann metric tensor, inner product and norm
$[\ ], \Gamma$	Christoffel symbols of the first and second kind.
	connexion on M compatible with g
$\mathfrak{X}(M)$	module of vectorfields on M
$\mathfrak{F}(M)$	module of real valued functions on M
Ω	volume on M
$\mathrm{div}_\Omega X$	divergence of a vectorfield wrt a volume
$\dot{c} = v$	given a curve c in M $v = \dot{c} = Tc(t,1)$ is the velocity.
$\frac{DX}{dt}$	given a vectorfield X along a curve c then $\frac{DX}{dt} = {}_{v}X$ is the covariant derivative of X along c.
L_X	Lie derivative wrt a vectorfield.
T, R	Torsion and Curvature tensors.
s	parameterised surface $s : I \times I \quad M : (u,v) \quad s(u,v)$
$\frac{\partial s}{\partial u}\ \frac{\partial s}{\partial v}$	variation vectorfields of the surface
T	Tangent functor.

All the above can be found in Spivak(25 & 26)

F^rM	the bundle of orthonormal r-frames
G	the induced Riemann metric on T^1M or F^rM
$\lvert\lvert\lvert\ \rvert\rvert\rvert, \langle\langle\ \rangle\rangle$	the induced inner product and norm given by G
$[\![\,,]\!], \Gamma\!\!\Gamma$	the induced Christoffel symbols of G

$(F^rU, F^r\varphi)$ induced charts on F^rM.

All the above can be found in Appendix A

A	matrix
A^T	transpose
$\bar{A}^1$	inverse
$\bar{A}^T$	$(\bar{A}^1)^T$
$\frac{dA}{dt}, \int A(u)du$	differential and integral of a matrix performed component-wise.

$$\begin{bmatrix} O & -I_n \\ I_n & O \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}' = \begin{bmatrix} C & A \\ A^T & I \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \quad \text{Hamiltonian equation}$$

$W(U_1, U_2)$ the Wronskian of solutions $(U_1, V_1), (U_2 V_2)$ of above.

The above can be found in Hartman(12) or Copell (6)

T	Kinetic Energy function.
V	Potential energy function.
L	Lagrangian
FL	Fibre derivative, Legendre transform.
T^*M	Cotangent bundle.
j	Natural isomorphism : $TM \rightarrow T^*M$
Ω_M	Natural volume on T^*M
$\Omega^\flat$	induced map $TM \rightarrow T^*M$
$\Omega^\#$	induced map $T^*M \rightarrow TM$

A	Action
E	Total Energy
X_E	induced vectorfield on TM.

All the above can be found in Abraham (1)

Bibliography.

We list the references used in this thesis, together with some others, which have been found useful in the period of preparation of this work. Those designated with ** are references from which results have been used, those with * references from which basic ideas and definitions have been used.

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